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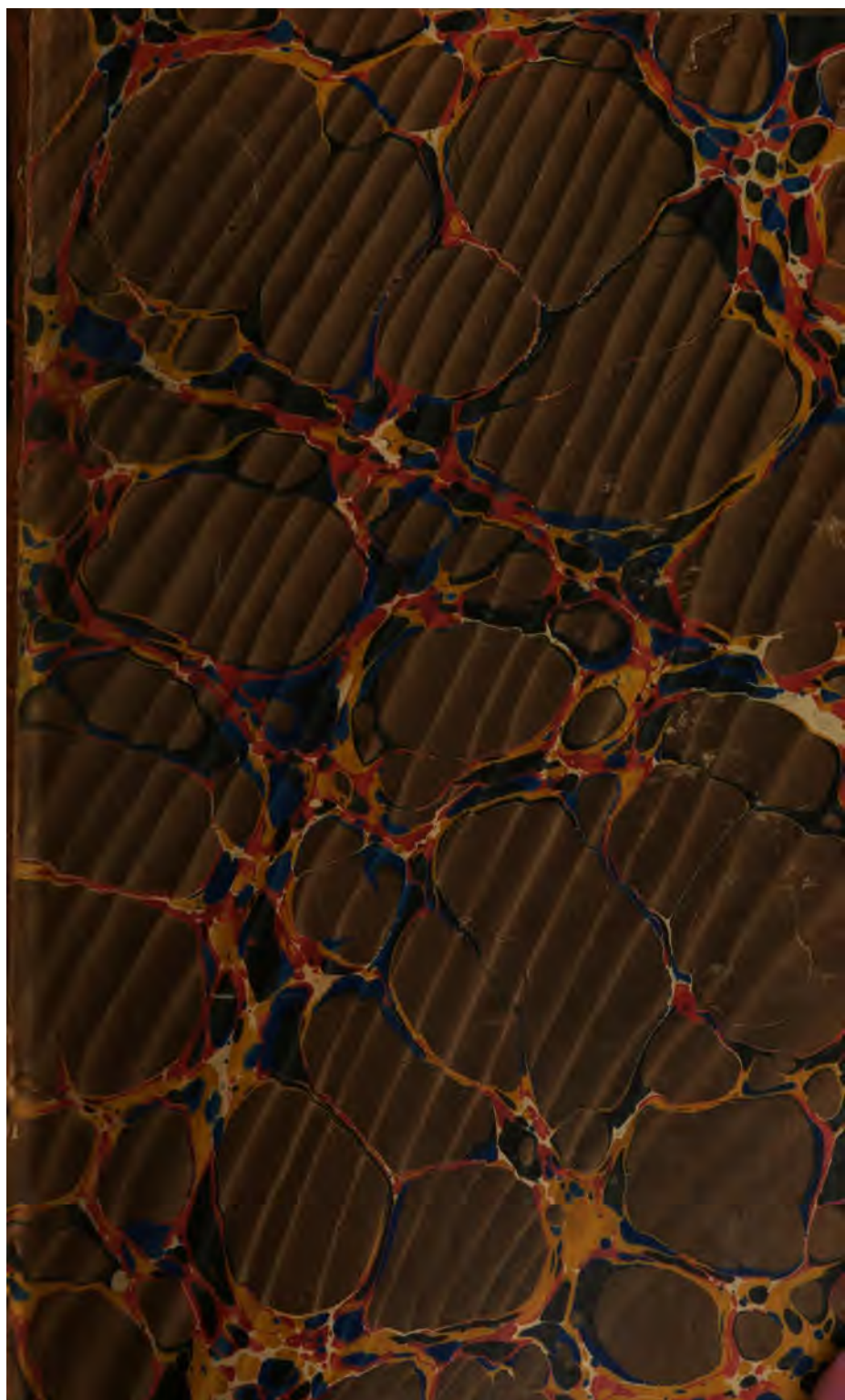
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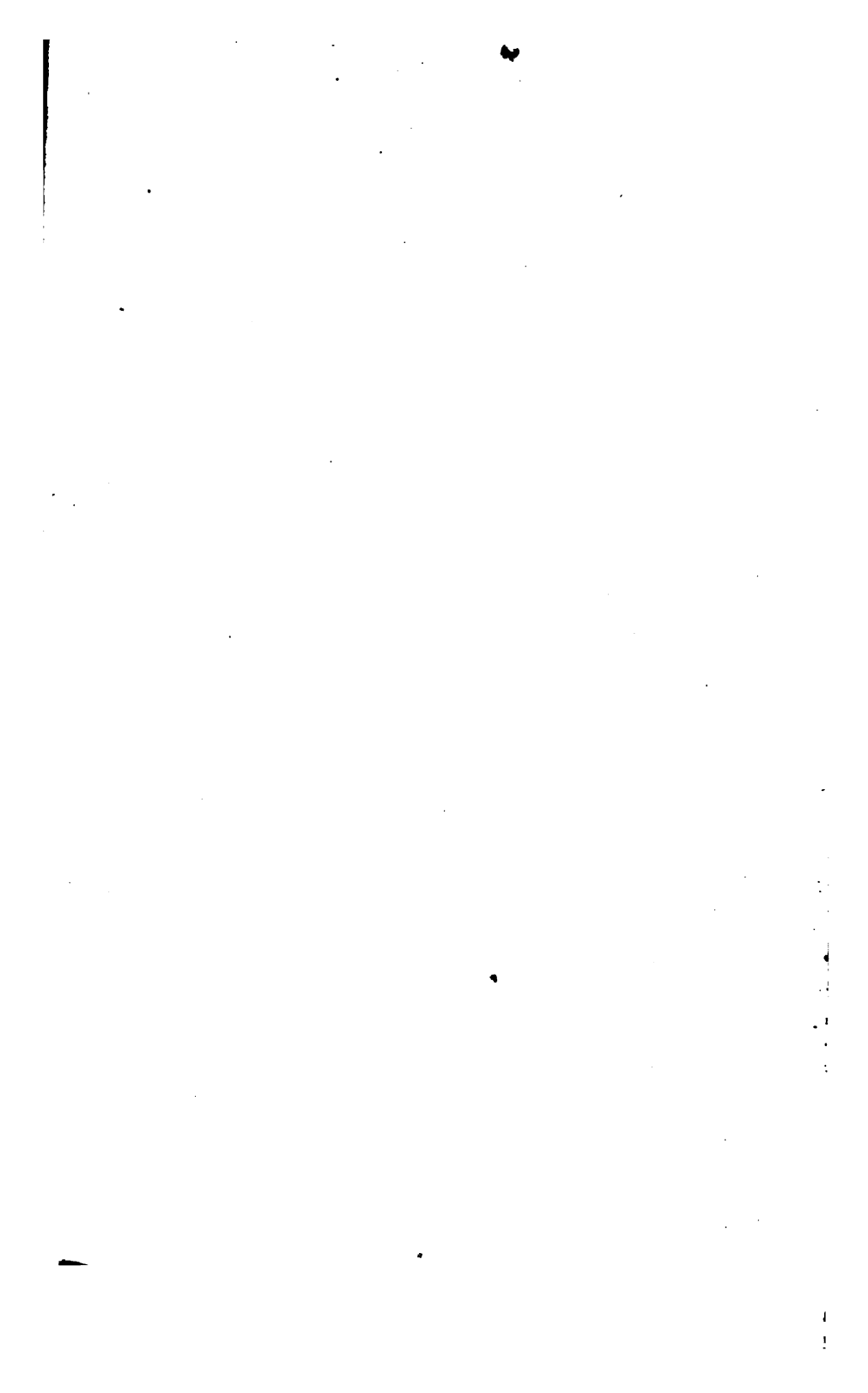
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NEW SERIES

OF THE

MATHEMATICAL

REPOSITORY.

By THOMAS LEYBOURN,
Of the Royal Military College.

VOL. I.

London :

PRINTED AND SOLD BY W. GLENDINNING, NO. 25,
HATTON GARDEN.

1806



Hist. of Science
Bowen
 10-29-31
 24455

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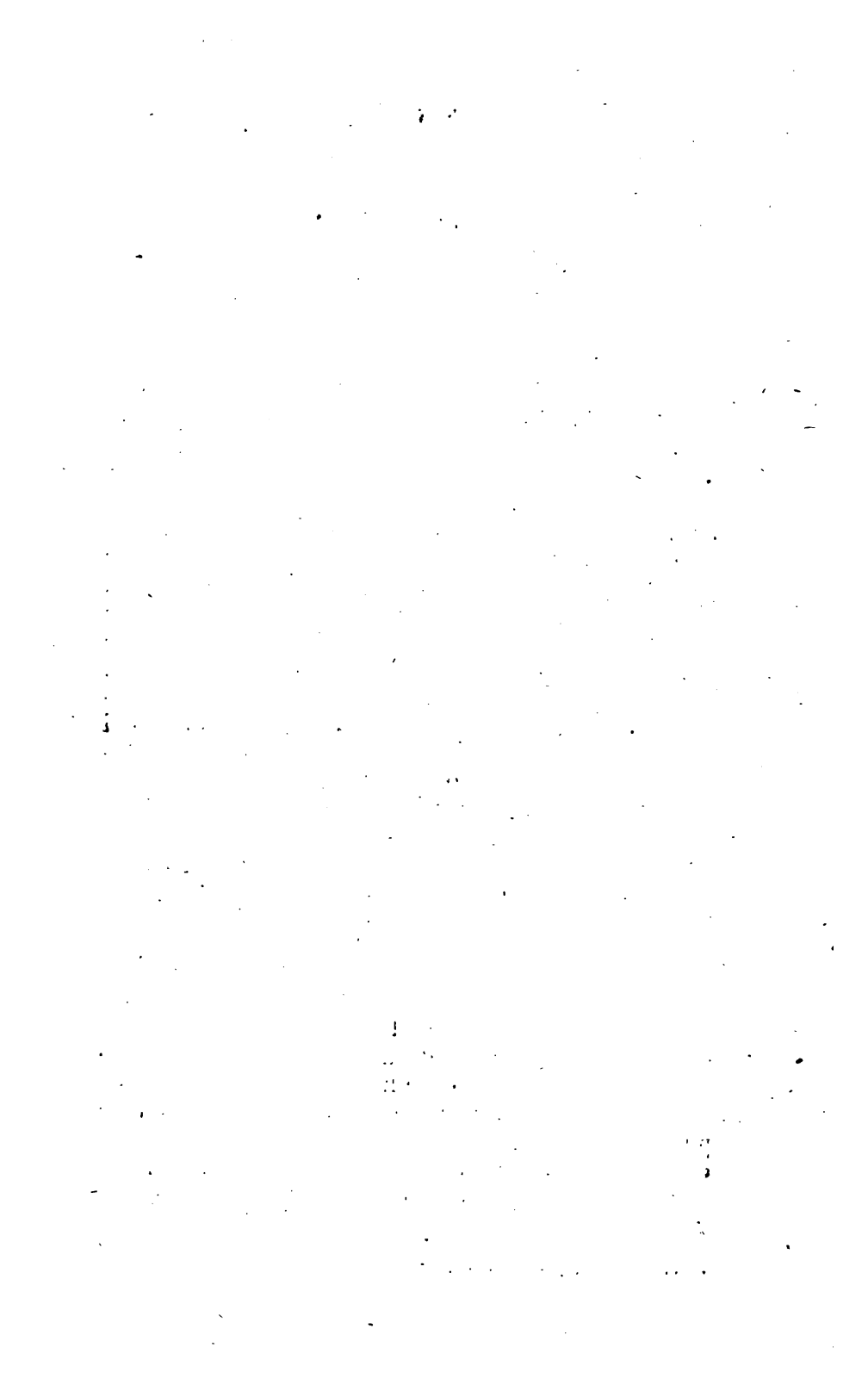
The first volume of the New Series of the Mathematical Repository being now compleated, it may be proper to state briefly its plan. The volume consists of three distinct parts. The first contains one hundred and twenty Questions, in almost every branch of the Mathematics. These are, with a few exceptions, entirely new, and each Question has at least one, but in general two or more answers, furnished by different ingenious Mathematicians.

The utility of this part of the work will be readily admitted when it is considered, that almost all the improvements which the Mathematics have received, have originated in the exertions made to resolve particular problems, such as that of the *trisection of an angle*, the *duplication of the cube*, &c. among the ancients; also the various *isoperimetrical problems*, and above all, the problem of the *three bodies* among the moderns. We believe also, that most Mathematicians will confess how much their talents have been cultivated and their knowledge improved, by resolving problems, such as are proposed in this volume.

- The second part consists of Original Essays on Mathematical Subjects. This department affords an opportunity to Mathematicians to publish such papers as may be judged worthy of preservation, but which are not of such magnitude as to form a distinct work.

The third and last part is composed of Mathematical Memoirs, extracted from Works of Eminence; which are chiefly selected from the Transactions of learned Societies, and such other works as are least likely to fall into the hands of general readers.

The first number of the second volume is now published; the remaining three numbers shall follow, one every six months, so that a second volume will be compleated in the course of eighteen months from this date.



NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

I. MATHEMATICAL QUESTIONS

To be Answered in Number II.

I. QUESTION 1, *by Tyro Philomatheticus, Hull.*

Given $\left\{ \begin{array}{l} x = 2z^2 + 8y \\ x + \frac{1}{3}y = y^3 + 2z \\ z^3 = y^2 + xy + \frac{1}{3}y + z^2 + 16z \end{array} \right\}$

To find x the year, y the month, and z the day of the month in which I was born.

II. QUESTION 2, *by Mr. I. H. Hearing, Adderbury.*

To find the length of that arc whose cosine is to its versed sine as the secant to the tangent, the radius being unity.

III. QUESTION 3, *by Mr. I. H. Hearing.*

A gentleman having in his garden a circular piece of ground, its diameter 12 yards, wishes his gardener to divide it into ten

equal little circular portions, in such a manner that each of them may touch the edge of the said piece of ground and one another, and in the middle to dig the greatest circular well for the convenience of watering, that the ground will possibly admit of, whose depth is to be thus ascertained; when he shall stand at the edge of the ground, his eye being then $5\frac{1}{2}$ feet above the level and the well filled with water, he may be just able to see the further edge of the bottom of it. *Query* the diameter of the circles and the depth of the well.

IV. QUESTION 4, by Mr. William Francis, *Teacher of the Mathematics and Land Surveyor, Maidenhead.*

What portion of the earth's surface might a person behold, if elevated so high above it as to lose $\frac{1}{3}$ of his weight, by the decrease of gravity, considering the earth a sphere of 3979 miles radius.

V. QUESTION 5, by Mr. John Lowe, *Attercliffe.*

To find three numbers in arithmetical progression such that the sum of every two of them diminished by the third shall be a square number.

VI. QUESTION 6, by Mr. T. Milner, *Sheffield.*

If two rings or circles of equal radii touch each other on their circumferences at P and one of them be made to revolve round the periphery of the other so that the area described by the point P may be equal to 2.68 square inches. *Query* the diameter of a circle equal to that of either ring?

VII. QUESTION 7, by Mr. John Whitley, *Attercliffe.*

Let FE, the diameter of a circle described about any plane triangle ACB, bisect the base AB perpendicularly in H, and let EP, FS be drawn perpendicular to AC: with the radii CH, CS and centre C describe the circles CQG and CSD meeting AC in G and S; and let S be the nearest point to C, then from P and G draw the tangents PQ, GD and the rectangle

PG, GD will be equal to the area of the triangle **ACB**. Required the demonstration.

VIII. QUESTION 8, *by Mr. I. T. M'Doneld.*

If on any given right line **AB**, there be taken any variable distance **AL**, and from **L**, in the same direction, any given invariable distance **LM**; and if with the centres **L, B** and radii **LA, BM** arcs be described, it is required to determine the nature of the curve, which is the locus of **P** the point of their intersection?

IX. QUESTION 9, *by Mr. John Wright, Norley.*

From two given points in the diameter of a given semi-circle, it is required, geometrically, to draw two right lines to meet each other at some point in the periphery so that the sum or difference of the respective rectangles under these two lines and two given ones (**M** and **N**) may be equal to a given magnitude.

X. QUESTION 10, *by Mr. William Smith, Liverpool.*

To determine the locus of the vertex of a triangle, the difference of whose sides and the radius of whose inscribed circle are given.

XI. QUESTION 11, *by Mr. John Wright, Norley.*

Given the base and vertical angle of a plane triangle to construct it, when the solid of the line bisecting the vertical angle into the square of the difference of the segments of the base made by the perpendicular is a maximum.

XII. QUESTION 12, *by Mr. James Cunliffe, Bolton.*

It is required to determine the *locus* of a point, from whence if perpendiculars be drawn to three straight lines given by positio

the sum of the squares of the said perpendiculars may be equal to a given magnitude.

XIII. QUESTION 13, *by* Scoticus.

Let ϕ and ψ be two arches such that $\sin. \phi + \cos. \psi = a$, and $\sin. \psi + \cos. \phi = b$: Required $\sin. \phi$ and $\cos. \phi$, also $\sin. \psi$ and $\cos. \psi$?

XIV. QUESTION 14, *by* Hypatia.

Required the value of $(1 + \frac{1}{n})^n$ when n is indefinitely great.

XV. QUESTION 15, *by* Analyticus.

Required a general expression for the area of a curve whose equation is

$$y \sqrt{1 - 3x^2} = \frac{a}{3 - x^2}.$$

XVI. QUESTION 16, *by* Amicus.

It is required to find the surface of an oblique cone the radius of the base being unity, the perpendicular from the vertex $\frac{1}{2}$ and the distance of that perpendicular from the centre of the base $\frac{1}{2}\sqrt{3}$.

XVII. QUESTION 17, *by* Geometricus.

A triangle being given by position, and two points being also given, it is required to draw lines through these points to intersect each other in one of the sides of the triangle and meet the other sides so as to cut off segments adjacent to given points in them, having to each other a given ratio.

XVIII. QUESTION 18, by Geometricus.

Let A, B, be two given points in AB a straight line given by position, and let C, D be two given points without that line, let CV, DV, be drawn meeting AB in F and G so that AF may have to BG a given ratio; It is required to determine the Locus of the point V.

XIX. QUESTION 19, by Mr. William Wallace, of the Royal Military College, Great Marlow, Bucks.

Let AB be the diameter of a circle, and FG a straight line perpendicular to the diameter, meeting it in C, draw any two lines CHD, CKE meeting the circle in H, D, also in K, E; join DK, EH, meeting the line perpendicular to the diameter in F and G, then FC is equal to CG: Required the demonstration?

XX. QUESTION 20, by Mr. William Smith, Liverpool.

Given the foci of an ellipsis to describe it so as to touch a circle given in position and magnitude,

XXI. QUESTION 21, by Mr. John Lowry.

Having given any number of points A, B, C, D, &c. in the periphery of a given circle it is required to find another point P such that joining PA, PB, PC, &c. the sum of the lines PA, PB, PC, &c. may be equal to a given line.

XXII. QUESTION 22, by Amicus.

It is well known that if a denote an arch of a circle, radius being unity, the tangent of that arch is expressed by the series $a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c.$ but it is not so generally known what is the scale of relation of the numeral coefficients of the terms; It is therefore required to determine that scale.

XXIII. QUESTION 23, *by Mr. James Cunliffe, Bolton.*

To find the least biquadrate number, that can be divided into four integral parts, such, that the sum of every two of them may be a square number.

XXIV. QUESTION 24, *by Jack Chance.*

A in $a + b$ relations, tells b lies; and B in $c + d$ relations, tells d of them false: What is the probability of the truth of a story in which they both agree?

XXV. QUESTION 25, *by Quinbus Flestrin.*

Given the base, the difference of the adjacent angles, and a line from the vertical angle to the middle of the base to construct the plane triangle.

XXVI. QUESTION 26, *by Mr. James Cunliffe.*

In a plane triangle there is given the perimeter, one of the angles, and the rectangle of the sides including that angle, to construct it.

XXVII. QUESTION 27, *by Mr. Thomas Bazley, Bolton.*

In any plane triangle whereof C is the vertex, O the centre of the inscribed circle, Q the centre of the circle touching the base and sides produced, D the point where the line bisecting the vertical angle cuts the base, and S the point where the said line cuts the circumscribing circle below the base. Then the points being joined it is $QD : CD :: SD : OD$; required the demonstration.

XXVIII. QUESTION 28, *by Mr. Bazley.*

Given the vertical angle, the radius of the inscribed circle, and the sum, difference, or ratio of the rectangle of the two

sides including the vertical angle and the rectangle of the two lines drawn from the extremes of the base to the centre of the inscribed circle; to construct the plane triangle.

XXIX. QUESTION 29, *by Mr. Bazley.*

There is an hemispherical bowl whose diameter at the brim is 20 inches, placed with its axis perpendicular to the horizon; an uniformly heavy thin inflexible rod or right line, whose length is 24 inches and weight 8 ounces, is sustained in equilibrio by resting on the edge of the brim at some distance from the upper extremity whilst the lower end rests against the concave surface of the bowl. It is required to determine by a general investigation, the position of the rod and the respective pressures at the two points of support.

XXX. PRIZE QUESTION 30, *by Philalethes Cantabrigienfis.*

Given $\frac{6xx^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = y$, in which equation x and y begin together to find the value of y when x becomes $= 10$.

The Solutions to these Questions must come to hand by the first day of February, 1804.



NEW SERIES
OF THE
MATHEMATICAL REPOSITORY.

II. MATHEMATICAL QUESTIONS

To be Answered in Number-III.

I. QUESTION 31, by Mr. T. Milner, Sheffield.

Being out with a quadrant, some of my pupils were desirous of knowing the height of the parish church spire, which rises to a considerable height in the middle of the town of Sheffield; to gratify their curiosity I found two stations in Church Street, distant from each other 70 yards, exactly on a level with the bottom of the spire, such that the altitude of the top of the spire at each station was equal to $35\frac{1}{2}$ degrees; and again, in the middle of this station line, the altitude of the spire's top was found to be $36\frac{1}{2}$ degrees. From hence it is required to find geometrically the height of the said spire, and its distance from the last station, supposing the quadrant to be held 5 feet from the ground at each station.

II. QUESTION 32, by Tyro Philomatheticus, Hull.

A gentleman has put ¹⁰⁰⁰ at 500l. at 3 per cent. per annum compound interest, and by agreement the interest is to be payable
VOL. I. B

half yearly. *Query* what sum must he take out of the flock at the end of the first half year, so that by continuing half yearly the like sum for 25 years, he shall then have received the whole principal and interest.

III. QUESTION 33, by Mr. I. H. Harding, *Adderbury.*

$$\text{Given } \left\{ \begin{array}{l} x + \frac{x}{1.3} + \frac{x}{2.3} + \frac{x}{2.5} + \frac{x}{3.5} + \frac{x}{3.7} + \&c. \text{ ad infin.} = 30y \\ \frac{1}{3} + y + 2y^2 + \frac{1}{3}y^3 + 5y^4 + \&c. \text{ ad infin.} = \frac{1}{10}x \\ \text{to find } x \text{ and } y. \end{array} \right.$$

IV. QUESTION 34, by Mr. William Francis, *Maidenhead.*

A circular piece of ground containing 220 acres is to be divided between three persons A, B, C, whose shares are to be as 5, 6, 7. *Query* the shares, by construction, so that the boundaries may be equal?

V. QUESTION 35, by Merlin.

To cut 5 equal and similar parts from 5 equal squares, so that the 10 pieces when joined together shall form a square.

VI. QUESTION 36, by Mr. I. T. M'Doneld.

Given the sum of the base and one of the sides, and the excess of the base above the other side, to construct the plane triangle when its area is either a given space or a maximum.

VII. QUESTION 37, by Mr. James Whalley, *Bolton-la-Moors.*

Having a circle and a triangle ABC, both given in magnitude and position; it is required to draw a tangent to the circle, cutting BC in P and AB in Q, so that drawing PR parallel to AB, the sum, difference, or ratio of the two triangles PQB and PRC may be given.

VIII. QUESTION 38, *by Mr. William Smith, Liverpool.*

Let AB be the axis of a conic section from B draw BD to meet the curve in D, demit DC perpendicular to AB, and produce it from C till CP is in a given ratio to BD; the locus of the point P will be a conic section. Required the demonstration.

IX. QUESTION 39, *by Mr. Thomas Bazley, Bolton.*

From the given point P in the periphery of a given semicircle APR, it is required to draw the right line PB meeting the diameter AR in B so that BC being drawn perpendicular to AR meeting the periphery in C; $BC^2 - BP^2$ shall be of a given magnitude.

X. QUESTION 40, *by Geometricus.*

Let two circles have the same centre, and let A, B be two given points in the circumference of one of them. It is required to inflect AC, BC to the circumference of the other circle so that the rectangle $AC \times CB$ may be a given magnitude.

XI. QUESTION 41, *by Hypatia.*

Three points being given by position, it is required to construct a triangle, the sides of which may be given in magnitude, and pass through the given points.

XII. QUESTION 42, *by Geometricus.*

A triangle being given by position, and two points being also given, it is required to draw lines through these points to intersect each other in one of the sides of the triangle and meet the other sides so as to cut off segments adjacent to given points in them such that the rectangle contained by these segments may be equal to a given space.

XIII. QUESTION 43, *by Sidrophel.*

If a triangle given in species have two of its angles upon straight lines given by position and the side adjacent to these angles passing through a given point, required the *Locus* of the angle opposite to that side.

XIV. QUESTION 44, *by Mr. John Wright, Norley.*

To find three square numbers in harmonical proportion such, that, if from each of them its respective root be taken, the remainders thence arising shall be rational square numbers.

XV. QUESTION 45, *by X. Y.*

It is required to investigate a rule for finding the area of any polygon whatever by means of its sides and angles,

XVI. QUESTION 46, *by Nauticus.*

Suppose a great circle to pass through a given point of the equator and make with it a given angle, it is required to find its projection on Mercators Chart,

XVII. QUESTION 47, *by Geometricus.*

Let A, B, be two given points in AB a straight line given by position, and let C, D be two given points without that line; let CV, DV be drawn, meeting AB in F and G, so that the rectangle $AF \times BG$ is given: It is required to determine the *Locus* of the point V.

XVIII. QUESTION 48, *by Mr. William Smith, Liverpool.*

Given the perimeter, the area, and the difference of the angles at the base of a plane triangle to construct it by the conic sections.

XIX. QUESTION 49, by Analyticus.

Required the area of a curve whose equation is

$$y \sqrt{1 + 4x^2 + x^4} = \frac{1 - x^2}{1 + x^2}$$

XX. QUESTION 50, by Amicus.

If a represent any arch of a circle, that arch is equal to the infinite product

$$\sin. a \times \sec. \frac{1}{2} a \times \sec. \frac{1}{4} a \times \sec. \frac{1}{8} a \times \sec. \frac{1}{16} a \times \sec.$$

As this very curious expression for a circular arch does not seem generally known, it is required to find its demonstration.

XXI. QUESTION 51, by Curiosus.

In a work entitled "*Hemisphaerium Dissectum*:" Authore Ricardo Albio, 4to. Rom. 1648, it is said that the convex surface of a scalene cone is equal to a circle whose radius is $\frac{1}{2} \sqrt{h^2 + r^2} + \frac{1}{2} S + \frac{1}{2} s$: where h is the cone's perpendicular height, r the radius of the base, S and s the greater and less slant heights. How is this rule derived?

* a mean proportion between r and S

XX.II QUESTION 52, by Aliquis.

Having given the base of a triangular pyramid, and the three plane angles at the vertex; to find the other dimensions?

XXIII. QUESTION 53, by Mr. Wm. Wallace, of the Royal Military College, Great Marlow, Bucks.

Let AB, AC be two straight lines given by position, and let DE be a line of a given length terminated by them at D and E , let DV, EV , be perpendiculars to the lines AB, AC , meeting in V . It is required to determine the *Locus* of the point V .

XXIV. QUESTION 54, *by Urfa Minor.*

Three straight lines being given by position, and three points being also given, it is required to describe a rectilinear figure, whose sides may pass through the given points, and whose angles may be upon the lines given by position.

XXV. QUESTION 55, *by Quinbus Flestrin.*

Given the base and the length of a line from the vertical angle to a given point therein, to construct the triangle when the vertical angle is a maximum.

XXVI. QUESTION 56, *by Mr. James Cunliffe.*

Divide the sum of three square numbers in arithmetical proportion into three other square numbers in arithmetical proportion, and give an example in whole numbers as small as may be.

XXVII. QUESTION 57, *by Mr. Cunliffe.*

Let a slender heavy inflexible rod AB, loaded with a weight be placed with its upper end B leaning against an inclined plane DE, and its lower end A upon the curve DA, situated in a vertical plane at right angles to the inclined plane. It is required to investigate the nature of the curve AD, so that the rod may rest in equilibrio when placed in any inclination.

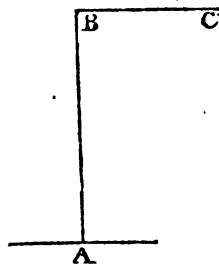
XXVIII. QUESTION 58, *by Mr. Bazley.*

AC, CB are two rods or right lines connected at C by a moveable joint, and whilst the end C of the rod BC revolves round

the fixed point B let the point A of the rod AC be carried along the right line AB passing through B ; it is required to determine the equation and area of the curve described by P, a given point in AC.

XXIX. QUESTION 59, by Sir Titus Triplicate.

Suppose a given cylindrical tube or pipe ABC, whereof AB is perpendicular, and BC parallel to the horizon to revolve round AB as an axis with a given angular velocity ; the lower end A standing in water, whose surface is at A, and the whole tube being previously filled ; it is required to determine the velocity with which the water will issue from the extremity C.



XXX. PRIZE QUESTION 60, by James Ivory, Esq.

THEOREM. Let n be any integer number, and let m be another integer number less than n : then

$0 = n^m - n \cdot (n-1)^m + n \cdot \frac{n-1}{2} \cdot (n-2)^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot (n-3)^m +$
 &c. the coefficients being the same with the coefficients of the binomial quantity $(a-b)^n$, and the progression, n^m , $(n-1)^m$, $(n-2)^m$ to be continued till we arrive at 1^m . Required the investigation.

The Solutions to these Questions must come to hand by the 1st of June, 1804.



NEW SERIES
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III. MATHEMATICAL QUESTIONS

To be Answered in Number IV.

I. QUESTION 61, by Mr. THOMAS SQUIRE, *Baldock.*

A tree growing upon the side of a hill which rises due North at an angle of 30° , had the upper part blown off 12 feet from the ground by a strong W. S. W. wind. It is required to find the height of the tree supposing it to stand perpendicular to the horizon, and the top (before the other part left the tree) to strike the ground 40 feet from its bottom.

II. QUESTION 62, by Mr. JOHN CAVILL, *Beighton.*

In any plane triangle it will be as the base is to the difference between the sum of the sides and base, so is the rectangle under half the perimeter and that part of the diameter of the circumscribing circle which is perpendicular to the base, and falls below it, to the area of the triangle. Required a demonstration.

III. QUESTION 63, by HYPATIA.

A circular ring revolves on the inside of another exactly double its size; required the locus of a given point in its circumference.

IV. QUESTION 64, *by Mr. THOMAS HOPPER, Manchester.*

Admitting the latitude of Manchester to be $53^{\circ} 25' N$. It is required to find the situation of two other places on the surface of the globe, such, that when it is noon at Manchester on the third day of March, 1805, the sun's altitude may be equal at all the three places; and the first place from Manchester may bear East-South-East, and the second place from the first West-North-West.

V. QUESTION 65, *by ARITHMETICUS.*

THEOREM. Let N be any number whatever, and let $a =$ the difference of N , and the next greater square number, and $b =$ the difference of N , and the next less square number. Then $N - a \times b$ will be always a square number.

VI. QUESTION 66, *by Mr. S. WADE.*

At what time in latitude $50^{\circ} 31'$ (the sun being in the equinox) will the shadow of the stile on the plane of a direct west vertical sun-dial be a minimum.

VII. QUESTION 67, *by Mr. BENJAMIN BEVAN.*

In a plane triangle, let w be the centre of a circle passing through x , y , and z ; then will $Cw = Cc$, and be in the same right line; and $wx = wy = wz = \frac{1}{2}R$, or the diameter of the circumscribing circle; where x , y , z , &c. represent the same points and lines as they denote in the Synopsis of Data for the Construction of Triangles.

VIII. QUESTION 68, *by ALCOR.*

Let A and B be two given points, CD , CE two straight lines given by position, let AF , BF be inflected to F any point

point in one of the given lines, and let BF meet CD in G, draw GH parallel to AB meeting AF in H, then H will be in a straight line given by position. Required the demonstration.

IX. QUESTION 69, *by* GEOMETRICUS.

Four straight lines are given by position, it is required to describe a triangle whose angles may be upon three of the lines, and whose sides may make given angles with the remaining line.

X. QUESTION 70, *by the Rev. L. EVANS, Royal Academy, Woolwich.*

It is required to bisect an arch of a circle by the description of circles only.

XI. QUESTION 71, *by* WHACUM.

Let AB be a given straight line, and P a given point without it, let CPD be drawn meeting AB in C, and let CP be to PD as AC to CB. Required the locus of the point D.

XII. QUESTION 72, *by* BOMBELLI.

To divide $\frac{1}{2}$ into 5 parts such, that either part added to the cube of $\frac{1}{2}$ shall be a square.

XIII. QUESTION 73, *by* HYPATIA.

When the base of a triangle is given, and one of the angles at the base is double the other; what is the locus of the vertex.

**XIV. QUESTION 74, by Mr. GEORGE SANDERSON,
London.**

Determine the value of $\frac{\text{cof. } \phi}{\text{cof. } \psi}$, when $\frac{\text{fin. } \phi}{\text{fin. } \psi} = n$, a given number, and $\phi + \psi = \alpha$, a given arch.

XV. QUESTION 75, by NAUTICUS.

Required the rectification of the line which is the projection of a great circle passing through a given point of the equator and making with it a given angle in Mercator's Chart.

XVI. QUESTION 76, by THEODOSIUS.

Having given the base of a spherical triangle upon the surface of the sphere and its area; required the locus of its vertex.

XVII. QUESTION 77, by Mr. JOHN WRIGHT, Norley.

Required the fluent of $\sqrt{\left[\frac{1+x^6}{(1+x^2)^5}\right]}$, with the assistance of the conic sections.

XVIII. QUESTION 78, by URSA MINOR.

It is required to demonstrate, independently of fluxions, that when an arch of a circle is small, twice the sine + the tangent = three times the arch nearly.

XIX. QUESTION 79, by Mr. JOHN WRIGHT.

Given the difference of the angles at the base, and the length of a line drawn from the vertex to the middle of the base to construct

construct the triangle when the solid of the perimeter, the difference between the sum of the sides and the base, and the perpendicular let fall from the vertical angle upon the base is a maximum.

XX. QUESTION 80, by Dr. MICHAEL STUART O'RIORDAN, *Professor of Mathematics, &c. in the College of Carlow, Ireland.*

Suppose a quantity of steel filings to be disposed uniformly in a curve-like figure, on an horizontal plane, and a magnetic globe indefinitely small be placed in its axis at the distance a . It is required to find the nature of the curve formed by the stratum, so that the globule shall reach it in the least time.

XXI. QUESTION 81, by Mr. THOMAS BAZLEY, *Bolton.*

It is required to determine the figure of an orifice to be cut in the vertical side of a vessel or dam, whose water is always of the same constant height; of such a nature, that the quantity of water issuing below the shuttle shall be always as the height of the opening, and that when the whole orifice is open q cubic feet shall issue per second; the top of the orifice being n , and the bottom m feet, below the surface.

XXII. QUESTION 82, by Mr. W. WALLACE.

Given $\left\{ \begin{array}{l} \cos. \phi + \cos. \psi = a \\ \cos. 5\phi + \cos. 5\psi = b \end{array} \right\}$ To find $\cos. \phi$ and $\cos. \psi$.

XXIII. QUESTION 83, by Mr. J. IVORY, *R. M. College.*

Let A and B be two points without a circle of which O is the centre, and let $BO \times OC =$ square of radius, and $AO \times OD =$ square of radius; and let BD and AC meet in E. Then, if the chord FG be drawn through E, and AF, FB, BG, AG be joined, $AF \times FB : AG \times GB :: FE : EG$. Required the demonstration.

XXIV.

XXIV. QUESTION 84, *by Mr. BENJAMIN BEVAN,*
Leighton Buzzard.

Let ABCD (fig. 50, pl. 2.) be the vertical section of a wall of Masonry, terminating at the top in the smooth curved plane AB, also let AE be a straight plane connected to the curve by a moveable joint or hinge at A, let GH be the surface of water in a reservoir, and FI a cylinder of less specific gravity than water connected by the rod IE and moveable joint E to the straight plane, the cylinder being confined to a perpendicular motion, or to move freely up and down in the fluid in which it floats; let K and L be two heavy rollers of the same weights and diameters, and suppose them connected by the inflexible rod KL, considered as without weight. It is required to determine the nature of the curve AB, such that the rollers may be sustained in equilibrio in any position of the rod KL.

XXV. QUESTION 85, *by Dr. O'RIORDAN.*

Admit a whipping top, formed by the rotation of a semi-curve about its axis whose equation is $ax^2 = y^3$; to determine at what distance from the vertex (or spinning point) a boy may strike it with a single lash, in a direction parallel to the horizon, that it may acquire the swiftest motion possible.

XXVI. QUESTION 86, *by SCOTICUS.*

If four straight lines intersect each other and form four triangles, circles described about each of these triangles will pass through one and the same point. Required the demonstration.

XXVII. QUESTION 87, *by Mr. JAMES NORTH.*

From the three angular points of a given plane triangle to draw three right lines meeting in the circumference of a given circle.

circle so as to have a given ratio to one another, the plane of the circle being perpendicular to the plane of the triangle.

XXVIII. QUESTION 88, *by Mr. JAMES IVORY.*

Let there be an ellipse of which AB is the greater and ED the less axis and centre C. Take CH in either axis equal to the sum or difference of the two semi-axes CB and CD, and upon CH let a circle be described; through B, the extremity of the axis, draw a straight line to cut the circle in F and G, and through F and G draw two diameters of the ellipse PC and QC. Then, the line FG will be equal to the sum of the semi-diameters CQ and CP in the one case and to their difference in the other; and if any line LK, equal to FG, be interposed between the diameters CP and CQ, and in LK (or in LK produced beyond the point K when CH is equal to the difference of the two axes) LM be taken equal to CP, the point M will be in the periphery of the ellipse. Required a demonstration,

XXIX. QUESTION 89, *by PHILOMATHES OXONIENSIS.*

Required the fluent of $\frac{x^{3/2}}{(1 + \sqrt{x})^3}$, generated while x increases from 0 to 2; and also the fluent of $\frac{xx^3}{(x^3 - 1)^{3/2}}$, generated while x increases from $\frac{1}{8}$ to 2.

XXX. PRIZE QUESTION 90, *by EDINBURGENSIS.*

Suppose a horse to move with a given velocity in the circumference of a given circle. Required the angle which the body of the horse will make with the horizon by leaning towards the centre.



The Solutions to these Questions must come to hand by the First of January, 1805.



NEW SERIES

OF THE

MATHEMATICAL REPOSITORY.

IV. MATHEMATICAL QUESTIONS,

To be Answered in Number V.

I. QUESTION 91, *by Capt. GEO. GORRY.*

Passing along a straight and level road, near a very lofty tower, and on the same horizontal plane with its base, I wished to ascertain its height; but being prevented by fences from quitting the road, and having no instrument but one for taking vertical angles, I proceeded thus: at a convenient point on the road, I found the angle of elevation of the top of the tower, $30^{\circ} 48'$; then measuring 60 yards along the road, I found a second angle of elevation $40^{\circ} 33'$; at the end of another 60 yards in the same direction, I intended taking a third angle of elevation, but the two were hidden by a high wall; I therefore measured on to 72 yards from the second station, and there found the angle of elevation of the top of the tower, $50^{\circ} 23'$. From these data, the height of the tower, and its nearest distance from each of the three stations, are required?

II. QUESTION 92, *by NEGOTIATOR.*

A sells goods to B, on condition to allow 10 per cent. discount, if he be paid in six months.—What discount ought A to allow, if payment be made him in two months;—interest of money being 5 per cent. per annum, simple interest?

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III.

III. QUESTION 93, by Mr. WM. SIMPSON, *Bolton.*

If a circle be described through the two foci of an ellipse, and any point in the conjugate axis produced; then a right line being drawn from that point, to one of the points where the circle cuts the ellipse, will be a tangent at that point. Required the demonstration?

IV. QUESTION 94, by CENTROBARICOS.

Suppose a cone, whose axis is 10, and diameter 8 inches, be made to vibrate, its vertex being the point of suspension; it is required to ascertain at what distance from its vertex, *on the flant side*, it must strike an immoveable obstacle, so that all its angular motions shall be destroyed, or that its point of suspension shall not be affected by the stroke?

V. QUESTION 95, by DIOPHANTUS.

It is required to find three rational numbers, x , y , and z , such that the formula

$x^2 + y^2 + 2z$, $x^2 + z^2 + 2y^2$ and $y^2 + z^2 + 2x^2$,
may be all squares?

VI. QUESTION 96, by Mr. JOHN DAWES, *Birmingham.*

In Mr. Watt's Steam engine, the reciprocating motion of the engine is converted into a circular motion, by a peculiar contrivance, consisting of two equal wheels, that work into one another: it is required to demonstrate that the central-wheel will turn round twice for every revolution of the Planet-wheel about it, or for every complete stroke of the engine; and also, to define what will be the motion of the central-wheel when it is not equal to the Planet-wheel, but in any given proportion to it?

VII. QUESTION 97, from *Geometrie Descriptive*, by MONGE.

If four unequal spheres be any how situated in space, and every two of them be circumscribed by a cone, so that its vertex may be on the same side of the centres of the spheres; the vertices of the fix cones thus constituted shall be at the intersections of four straight lines situated in a plane: and if every two of the spheres

spheres be circumscribed by another cone, having its vertex between the centres of the spheres; the vertices of these cones, taken three by three, shall be in the same plane as three of the former. Required the demonstration ?

The next Ten Questions, taken from the Cambridge University Calendar, are selected from the SENATE HOUSE PROBLEMS, proposed to Candidates for the Degree of B.A. in January, 1804.

VIII. QUESTION 98, *from the CAMB. UN. CAL.*

Three known rectilinear objects, placed contiguous, and in the same right line, appeared to the eye of a spectator of the same length; find his position ?

IX. QUESTION 99, *from the CAMB. UN. CAL.*

Shew how the circumference of an ellipse may be computed, of which the excentricity is .99, the semi-axis major being 1 ?

X. QUESTION 100, *from the CAMB. UN. CAL.*

Shew that the rectification of the hyperbola may be deduced from that of the ellipse.

XI. QUESTION 101, *from the CAMB. UN. CAL.*

In the lemniscata, whose equation is $(x^2 + y^2)^2 = x^2 - y^2$, it is required to assign two arcs equal to one another ?

XII. QUESTION 102, *from the CAMB. UN. CAL.*

On a given straight line, perpendicular to the horizon, describe a circle, and draw a tangent at its higher extremity; if any point be assumed in this tangent, and a straight line be drawn from it parallel to the diameter, cutting the circle in two points, and chords be drawn from those points perpendicular to the diameter, the times of describing the parts of this line, together with the times of describing the corresponding chords with the velocities acquired at their extremities shall be equal ?

XIII. QUESTION 103, *from the CAMB. UN. CAL.*

Let the compressive force be supposed, proportional to the square of the density, and the force of gravity inversely proportional to the square of the distance from the earth's centre; it is required to find the law of density in the atmosphere?

XIV. QUESTION 104, *from the CAMB. UN. CAL.*

A body revolving in a given circle, acted on by a force, which is inversely proportional to the square of the distance from the centre, in consequence of an impulse in the direction of its motion, begins to describe an orbit of a given excentricity; find the velocity communicated to it by the impulse?

XV. QUESTION 105, *from the CAMB. UN. CAL.*

Two bodies, whose weights are known, lying on a smooth horizontal plane, are connected by a flexible line passing through a small ring fixed at a given point between them; in this position a given velocity is communicated to one of them in a direction perpendicular to the line that joins their centres, and the other is made to move directly towards the ring: investigate the motion of the projected body, and find the angle described when the other body arrives at the ring?

XVI. QUESTION 106, *from the CAMB. UN. CAL.*

If a body, suspended by a string, oscillates through a quadrant (the extremity of the quadrant being the lowest point), the force stretching the string at the lowest point is three times that which is due to the weight of the body. Required proof?

XVII. QUESTION 107, *from the CAMB. UN. CAL.*

Let two weights fastened to two equal strings (l, l'), move in a vertical plane, the one oscillating, the other revolving; and at the lowest points let the velocities due to the heights h, h' , be such, that

$$\frac{h}{2l} = \frac{2}{h'} = m^2, \text{ then the time of the revolution of one weight is to}$$

the time of oscillation of the other as m to 1. Required proof?

XVIII.

XVIII. QUESTION 108, by Mr. JAMES CUNLIFFE.

It is required to find two square numbers, a^2 , b^2 , and a certain number n , which being either added to, or subtracted from either of the squares a^2 , b^2 , the sums and differences thence arising shall be rational squares?

XIX. QUESTION 109, by MARLOVIENSIS.

In Counsellor Cooper's Letters on the Slave Trade, we meet with the following question, and an algebraic *formula* annexed, exhibiting a general solution thereto.

"It is required to shew the average annual import of Negro Slaves (and the whole number imported), having given the annual decrement, or proportion of deaths to the whole stock, and the number of existing stock at the end of a given term?

"General Solution.

$$\frac{ba^n}{(a-1) \times (a^n - (a-1)^n)} = \text{the average annual import.}$$

"Where n denotes the given interval in years.

b , the stock existing at the end of that term.

a , the reciprocal of the annual decrement.

"Example. The first export of Negro Slaves from Africa was in the year 1503: and in 1770, being an interval of 276 years, it was found, from a pretty exact estimate, that the number of Negro Slaves then existing in the European, &c. powers, was five millions: the proportion of deaths among them was also determined from a series of observations to be about 1 in 20. These numbers substituted in the above formula, give

the average annual import 263, 158

the total imported (in 276 years) . . . 70, 263, 186!"

And, we may add, if estimated to the present time, we behold near 80 millions of the human species the innocent victims of avarice, cruelty, and wretchedness, by this diabolical traffic! An Investigation of the above *Formula* is required?

XX. QUESTION 110, by Mr. WM. SMITH, *Liverpool*.

If a given plane triangle be placed, so that two of its angular points may always be on two straight lines given by position; it is required to determine the locus of the remaining angle?

XXI.

XXI. QUESTION 111, *by* PORUS.

Required to inscribe, geometrically, the maximum ellipsis in a given trapezium?

XXII. QUESTION 112, *by* PORUS.

If one end of a string be fixed to the extremity of a body, placed on a perfectly polished plane, and the other to a point in that plane, and any force be impressed upon that body, it is required to determine its motion?

XXIII. QUESTION 113, *by* SCOTICUS.

Let EF, a straight line given in magnitude, be interposed between AC, BC, two straight lines given in position; let G be a given point in that line, then it is known that the locus of G is a given ellipse. Draw ED, FD, perpendicular to the lines given by position meeting in D, and join DG, then DG shall be perpendicular to a tangent to the ellipse at G. Required the demonstration?

XXIV. QUESTION 114, *by* Mr. IVORY, *R. M. College.*

Let AB, CD, be two straight lines, cutting one another in G, and terminating in an ellipse, of which F is one focus, and draw AF, FB; FC, FD: then will

$AF \times BG + BF \times AG : DG \times FC + FD \times GC :: AB : CD.$
Required the demonstration?

XXV. QUESTION 115, *by* Mr. WALLACE; *R. M. College.*

Required the locus of the angles of a parallelogram, formed by drawing tangents at the vertices of any two conjugate diameters of a given ellipse?

XXVI. QUESTION 116, *by* MARLOVIENSIS.

Suppose the river Thames, opposite *Canvey* Island, situated due east and west, to be two miles broad at high-water, and one
at

at low, and that the tide ebbs, at certain times, six hours, the breadth decreasing from each side equal spaces in equal times. Suppose also the velocity of the water in the middle of the stream, to increase as the time for the first three hours, when it then moves six knots *per* hour; and that the velocity in the middle is to that in any other part, as the semi-breadth of the water, to the nearest distance of that part to the adjacent sides. These granted, what course, and how many knots *per* hour, must a boat sail from the Kentish shore directly opposite *Bowley*, on the Essex side, at high water, to make *Scar House*, on *Canvey Island*, after a three hours trip, which is exactly three miles to the eastward of the place sailed from?

XXVII. QUESTION 117, by HYPATIA.

Three circles being given by position, it is required to draw a tangent to each of them, so as to intersect in the same point, and have the segments between that point and the points of contact in a given ratio to each other?

XXVIII. QUESTION 118, by A. B.

Let C be a body of a given weight, (suppose 875 grains), placed on a smooth and polished horizontal plane; let a thread or string perfectly unelastic and inextensible be attached to the body C, and after passing through two small rings, A and B, (so as to act on the body in a direction parallel to the horizontal plane,) fall down in a vertical direction: to the extremities of the thread, let two equal weights, P and Q (suppose each = 3500 grains), be fastened; then, it is required to define the motion of the body C, and to ascertain its place at the end of any given time, supposing it to begin to move from a given place, equally distant from the two rings, abstracting from the weight of the string, friction, and the resistance of the air?

XXIX. QUESTION 119, by PHILOMATHES OXONIENSIS.

It is required to find the fluent of $x\sqrt{1 + \frac{1}{x^3}}$, generated, while x increases from $\frac{1}{10}$ to 10, and to give the work at large, bringing out the result true to four places of decimals, at least?

XXX.

XXX. PRIZE QUESTION 120, *by* EDINBURGENSIS.

A polygon being given in magnitude and position, it is required to describe a polygon of an equal number of sides, having its several angles on the several sides of the given polygon, and such that the perimeter shall be the least possible ?



Solutions to these Questions must come to hand (post paid,) by the first of August, 1805.

NEW SERIES
OF THE
MATHEMATICAL REPOSITORY.

VOL. I. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

Solutions to Questions proposed in Number I.

I. QUESTION 1, by TYRO PHILOMATHETICUS, *Hull.*

$$\text{Given } \left\{ \begin{array}{l} x = 2z^2 + 8y \\ x + \frac{1}{3}y = y^3 + 2z \\ z^3 = y^3 + xy + \frac{1}{3}y + z^2 + 16z \end{array} \right\}$$

To find x the year, y the month, and z the day of the month in which I was born.

SOLUTION.

Questions of this nature, being confined to whole numbers, are generally most easily resolved by trial; accordingly here, y cannot be greater than 12, and by the second and third equations, it must be 3 or some multiple thereof: hence the least value of y , that gives z a whole number, is 12; then $z = 29$ and $x = 1778$. Therefore the Proposer was born on the 29th day of December, 1778.

Thus the Question was answered by Messrs. Cavill, Codling, Collins, Cornthwaite, Hopper, Mabbott, Marrat, Merones Minor, Morley, Pickering, Scoticus, and Scurr.

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(A)

II. QUES

II. QUESTION 2, by Mr. I. H. HEARDING, *Adderbury.*

To find the length of that arc whose cosine is to its versed sine as the secant to the tangent, the radius being unity.

FIRST SOLUTION, by Mr. JOHN DAWES, *Birmingham.*

Fig. 1. Pl. 1. Draw the perpendiculars FG, DE; then will all the triangles be similar, and $AF : FG :: AC : CB$; but by the question $AF : FB :: AC : CB$, therefore, $FG = FB = DE$, consequently the triangles DFG, CDE are equal in all respects; therefore, $CD = DF$; or the arc is that of which the secant minus radius (1) is equal to the sine, that is, $57^{\circ} 4'$ nearly.

SECOND SOLUTION, by the Rev. THOMAS SCURR, *Hexham.*

Let $v =$ versed sine, then will $\sqrt{(2v - v^2)}$ express the sine, $1 - v$ the cosine, $\frac{\sqrt{(2v - v^2)}}{1 - v}$ the tangent, and $\frac{1}{1 - v}$ the secant. Now, by the question, cosine : versed sine :: secant : tangent, that is, $1 - v : v :: \frac{1}{1 - v} : \frac{\sqrt{(2v - v^2)}}{1 - v}$; multiplying means and extremes we have $v = (1 - v) \sqrt{(2v - v^2)}$, and from this equation v is easily found $= .45633$ the versed sine of $57^{\circ} 4'$ nearly, the length of which is .996.

THIRD SOLUTION, by Mr. P. THOMPSON, *Frieston.*

Let ϕ be the arch required, then putting radius $= 1$;

cof. $\phi : 1 - \text{cof. } \phi :: \text{sec. } \phi : \text{tan. } \phi :: 1 : \text{fin. } \phi$

therefore, cof. ϕ fin. $\phi = 1 - \text{cof. } \phi$

and fin. $\phi = \frac{1}{\text{cof. } \phi} - 1 = \text{sec. } \phi - 1$.

Hence we have only to search the tables till we find an arch such that the number expressing the sine is the same as the decimal part of the number expressing the secant; thus we find $\phi = 57^{\circ} 4'$ nearly.

FOURTH

FOURTH SOLUTION, *by Mr. GEORGE PICKERING, at the Rev. JAMES LINDSAY's Academy, Newington-green.*

By the question $\text{cof.} : \text{versed f.} :: \text{sec.} : \tan,$

that is, $\text{cof.} : 1 - \text{cof.} :: \text{sec.} : \frac{\sin.}{\text{cof.}}$

or $\sin. = (1 - \text{cof.}) \times \text{sec.} = \text{sec.} - \text{sec.} \times \text{cof.} = \text{sec.} - 1.$

Hence, as above, the arch is found $= 57^{\circ} 4'$ nearly.

Solutions to this Question were also received from Messrs. Cavill, Codling, Collins, Harding, Hopper, Mabbott, Marrat, Merones Minor, Morley, Myers, and Scoticus.

III. QUESTION 3, *by Mr. I. H. HEARDING.*

A gentleman having in his garden a circular piece of ground, its diameter 12 yards, wishes his gardener to divide it into ten equal little circular portions, in such a manner that each of them may touch the edge of the said piece of ground and one another, and in the middle to dig the greatest circular well for the convenience of watering, that the ground will possibly admit of, whose depth is to be thus ascertained; when he shall stand at the edge of the ground, his eye being then $5\frac{1}{2}$ feet above the level and the well filled with water, he may be just able to see the further edge of the bottom of it. *Query* the diameter of the circles and the depth of the well,

FIRST SOLUTION, *by Mr. THOMAS BOOLE, London.*

CONSTRUCTION. Fig. 2. Pl. 1. Divide the given circumference into ten equal parts, by drawing the radii, OA, OB, OC, &c. join A, B, and produce OB to L, making $BL = \frac{1}{2}AB$; draw AL, and parallel to it, through B, draw Ba to meet Oa in a; with distances each equal to Aa, as radii describe the ten little circles, aAm, bBn, &c. and lastly, from the centre O, with the radius Om, equal to OA — Am, describe the circle Omnpqrstuvw, and the thing is done.

DEMONSTRATION. Join ab, bc, cd, &c. then, since by constr. Aa, Bb, Cc, &c. are all equal, Oa, Ob, Oc, &c. must likewise be all equal, and therefore ab parallel to AB, and because AL is also parallel to aB, the triangles ABL, abB, are similar;

(A 2)

therefore,

therefore, AB being equal to $2BL$, ab must also be $= 2bB$, and consequently the two little circles aAm , bBn , touch one another. Again, the angles aOb , bOc , cOd , &c. as likewise the sides aO , bO , cO , &c. being all equal, and ab being (as has been already proved) $= 2aA$, bc must be equal $2bB$, $cd = 2cC$, &c. and hence the circles a , b , c , &c. manifestly all touch one another, and each passes through a point in the periphery of the given circle by construction.

CALCULATION. In the triangle abB are given the ratio of ab to bB as 2 to 1, and the included angle $abb = 108^\circ$; to find the angle $aBb = 49^\circ 36' 50''$: also, in the triangle aOB we have given all the angles and the side $OB = 6$ yards; hence, we find $Oa = 4.583597$ yards, and $Aa = 1.416403$ yards. Then $6 - 1.416403 \times 2 = 3.167194$, and twice this $= 6.334388$ yards is the diameter of the inner circle, or of the well. Let $msXZ$ (Fig. 3. Pl. 1.) represent the well, $Am = 2.832806$ yards, the distance of the outer circle from the edge of the water, and $AS = 5\frac{1}{2}$ feet, the height of the eye; then $Sm = \sqrt{(Am^2 + AS^2)}$ is 10.1229 feet, and by similar triangles, as $Am : AS :: ms : sQ = 12.29846$, and $AS : Sm :: sQ : mQ = 22.6357$; also, as $Sm : \text{rad.} :: Am : \sin. \angle ASm$ or $ZmQ = 57^\circ 5' 23''$, the angle of incidence, its natural sine is $= .8395223$, and $\frac{1}{2}$ of this $= .6296417$ is the sine of the angle of refraction, answering to $39^\circ 1' 25'' = \angle ZmX$; then $57^\circ 5' 23'' - 39^\circ 1' 25'' = 18^\circ 3' 58'' = \angle XmQ$, and hence, as $\sin. \angle XmQ (= ZmX. : mQ :: \sin. \angle Xmq : QX = 11.14866$ feet, and consequently, $12.29846 + 11.14866 = 23.44712$ feet is the depth of the well.

SECOND SOLUTION, by the Rev. THOMAS SCURR.

Fig. 2. Pl. 1. Let the centres of the several circles be joined, and let $O\delta$ be drawn which will be perpendicular to ak ; also let the radius of the circular garden be denoted by r , and the radius of each of the required equal circles by x ; then will $O\delta = r - x$. Since, the joining of the centres of the several circles forms a decagon, in the isosceles triangle kOa , we have as $\sin. Oka : Oa :: \sin. kOa : 2k\delta = ka$, that is, in specie, as $\sin. b(72^\circ) : r - x :: \sin. c(36^\circ) : 2x$, or $2bx = rc - cx$; whence

$$x = \frac{rc}{2b + c} = 1.4164, \text{ and } 2x = 2.8328 \text{ yards the diameter of}$$

each of the equal circles. Hence, by proceeding as in the foregoing solution, the diameter and depth of the well are found equal to 6.3343 , and 7.8157 yards respectively.

THIRD SOLUTION, by Mr. JOHN CAVILL, *Brighton,*
near Sheffield.

Fig. 3. Pl. 1. Having found Am and ms , the diameter of one of the equal circles, and the diameter of the well, let AS be made perpendicular to Ams and equal to $5\frac{1}{2}$ feet, the height of the eye; produce smA to D , so that mD may be to sm as 4 to 3, let DC , mZ and sX be drawn parallel to AS and then through m , S , let $CSmQ$ be drawn meeting DC in C and sX in Q ; now make $mX = mC$, meeting sX in X and XZ parallel to ms for the bottom of the well. Hence, by calculation, sX or mZ , the depth of the well is found $= 7.8$ yards nearly.

Ingenious Solutions were also received from Messrs. Codling, Collins, Hearing, Hopper, Morley, Myers, Pickering, Squire, and Thompson,

IV. QUESTION 4, by Mr. WILLIAM FRANCIS, *Teacher of*
Mathematics and Land Surveyor, Maidenhead.

What portion of the earth's surface might a person behold, if elevated so high above it as to lose $\frac{1}{5}$ of his weight, by the decrease of gravity, considering the earth a sphere of 3979 miles radius.

FIRST SOLUTION, by Mr. THOMAS MYERS, *at the Rev. G.*
BATHIE'S School, Hammer Smith.

When a person is elevated so high above the earth's surface as to lose $\frac{1}{5}$ of his weight, it is evident that his weight upon the surface will be to his weight when elevated as 5 to 4, and the power of gravity being inversely as the square of the distance from the centre, it will be as $4 : 5 :: 3979^2 : \frac{5 \times 3979^2}{4}$, the square root of which is $3979 \times \frac{1}{4} \sqrt{5}$ the distance from the centre, from which deducting 3979 leaves 469.656 miles the height to which he must ascend. Hence, by the rules of mensuration, 10502188.84 square miles $= 6721400857.6$ acres, or part of the earth's surface seen by supposing him raised so high above the earth's surface as to lose $\frac{1}{5}$ of his weight.

SECOND

SECOND SOLUTION, *by Mr. WILLIAM MARRAT, Boston.*

Fig. 4. Pl. 1. Let ABC represent the earth, D its centre, DC the radius drawn perpendicular to the tangent EC, CFG a chord perpendicular to DE, and BE the height required; also suppose the person's weight at B to be = unity, the radius DB = DC = 3979 miles = a , and $p = 3.14159$, &c. Then, the force of gravity above the surface being inverfely as the fquare

of the diftance from the centre, it will be as $\frac{4}{5} : 1 :: a^2 :$

$\frac{5a^2}{4}$ = the fquare of the diftance from the centre of the earth ;

therefore $\sqrt{\frac{5a^2}{4}} = \frac{a}{2} \sqrt{5} = DE$, the diftance itfelf, and DE

— $DB = \frac{a}{2} \sqrt{5} - a = 469.656 = BE$, the height above the furface. Now, by fimilar triangles, $DE : DC :: DC : DF$,

i. e. $\frac{a}{2} \sqrt{5} : a :: a : \frac{2}{5} a \sqrt{5}$, and $a - \frac{2}{5} a \sqrt{5} = a \times (1 - \frac{2}{5} \sqrt{5})$ is the height of the fegment ; therefore, (by Prob. 10, pa. 99, of Dr. Hutton's Menfuration, Edit. 3.)

$2ap \times a \times (1 - \frac{2}{5} \sqrt{5}) = 2a^2 p \times (1 - \frac{2}{5} \sqrt{5})$
 $= 10502188.84$ fquare miles, is the portion of the furface that could be feen by a fpectator at E.

THIRD SOLUTION, *by MERONES MINOR.*

The force of gravity being inverfely as the fquare of the diftance from the earth's centre, putting $r = 3979$, $p = 3.14159$, &c. and x = the diftance required, we have

$\frac{1}{r^2} : \frac{1}{x^2} :: 1$, the force of gravity at the diftance $r : \frac{4}{5}$; hence,

$x^2 : r^2 :: 5 : 4$, or $x : r :: \sqrt{5} : 2$, and $x = \frac{r \sqrt{5}}{2}$.

Then, by Menfuration, the area of the vifible part of the furface is 10502188.84 fquare miles.

Other ingenious Solutions were given by Meffrs. Boole, Cavill, Codling, Collins, Hearing, Hopper, Morley, and Thompson.
 V. QUES.

V. QUESTION 5, by Mr. JOHN LOWE, *Attercliffe*.

To find three numbers in arithmetical progression such that the sum of every two of them diminished by the third shall be a square number.

FIRST SOLUTION, by Mr. J. I.

Let a, b, c be the three numbers required, then by the conditions of the question

$$\begin{aligned} a + c &= 2b \\ a + c - b &= x^2 \\ a + b - c &= y^2 \\ c + b - a &= z^2. \end{aligned}$$

Let b be exterminated from the two last equations by means of the first, and there will result

$$\frac{3}{2} a - \frac{1}{2} c = y^2$$

$$\frac{3}{2} c - \frac{1}{2} a = z^2$$

whence $a = \frac{1}{4}(3y^2 + z^2)$ and $c = \frac{1}{4}(3z^2 + y^2)$.

Combining the two first equations it will easily be found that $b = x^2$: and putting the values of a, b , and c that have been found in the first equation, we get

$$\begin{aligned} y^2 + z^2 &= 2x^2 \\ \text{or } \left(\frac{y+z}{2}\right)^2 + \left(\frac{y-z}{2}\right)^2 &= x^2. \end{aligned}$$

And this last equation is the only condition necessary; the numbers x, y , and z being in every other respect arbitrary or indeterminate. We may, therefore, assume for $\frac{y+z}{2}, \frac{y-z}{2}$ and x , any three numbers that measure the sides of a rational right-angled triangle, and hence innumerable solutions may be derived.

The least numbers that form a right-angled triangle are 4, 3, and 5: put $\frac{y+z}{2} = 4, \frac{y-z}{2} = 3$, and $x = 5$; then $y = 7$ and $z = 1$: whence

$$b =$$

$$b = x^2 = 25$$

$$a = \frac{1}{4}(3y^2 + z^2) = 37$$

$$c = \frac{1}{4}(3z^2 + y^2) = 13$$

And these are the least numbers that satisfy the question.

SECOND SOLUTION, *by Mr. JOHN WRIGHT, Norley.*

Let x, y and z denote the numbers sought. Then, since $x + y - z, x + z - y$, and $y + z - x$ is each to be a square number, let them be equated with a^2, b^2 , and c^2 respectively, and we shall thence obtain

$$x = \frac{a^2 + b^2}{2}, y = \frac{a^2 + c^2}{2}, \text{ and } z = \frac{b^2 + c^2}{2}.$$

Now per question $x + z = 2y$, that is, $\frac{a^2 + 2b^2 + c^2}{2} = a^2 + c^2$,

or $a^2 + c^2 = 2b^2$: hence must $a = 2rs + r^2 - s^2, b = r^2 + s^2$, and $c = 2rs + s^2 - r^2$, as is well known. These expressions for a, b , and c being written instead of them in the equations x, y , and z above, they become

$x = (r^2 + s^2)^2 + 2rs(r^2 - s^2), y = r^2 + s^2$ and $z = (r^2 + s^2)^2 + 2rs(s^2 - r^2)$, where r and s may be expounded by any two numbers taken at pleasure.

Ex. Gr. Take $r = 1$, and $s = 2$; then $x = 13, y = 25$, and $z = 37$.

THIRD SOLUTION, *by Mr. THOMAS BAZLEY, Bolton.*

Let $x^2 - y, x^2$ and $x^2 + y$ be the three numbers; then $x^2, x^2 - 2y$, and $x^2 + 2y$ are to be squares; the first is evidently so, and for the second assume

$$x^2 - 2y = x^2 - 2(2rx - 2r^2) = x^2 - 4rx + 4r^2 = (x - 2r)^2, \text{ a square.}$$

Now for the third $x^2 + 2y$, or $x^2 + 4rx - 4r^2$, assume it $= z^2$;

then $z^2 - x^2 = 4rx - 4r^2$, or $(z + x) \times (z - x) = 4r(x - r)$; therefore, by Mr. Leslie's Principle *.

$$(z + x) \times m(z - x) = 4rm(x - r), \text{ or } z + x = 4rm, \text{ and } m(z - x) = x - r$$

$$\text{whence } z = 4rm - x = \frac{x - r + mx}{m}; \text{ hence } x = \frac{4rm^2 + r}{2m + 1};$$

* See Mathematical Repository, Vol. I. Old Series.

and, consequently, $y = 2rx - 2r^2 = \frac{8r^2 m^2 + 2r^2}{2m + 1} - 2r^2$,

which is easily reduced to $\frac{4r^2(2m^2 - m)}{2m + 1}$; and, therefore, the

three numbers are,

$$\left(\frac{4rm^2 + r}{2m + 1}\right)^2 - \frac{4r^2(2m^2 - m)}{2m + 1}; \left(\frac{4rm^2 + r}{2m + 1}\right)^2 \& \left(\frac{4rm^2 + r}{2m + 1}\right)^2 + \frac{4r^2(2m^2 - m)}{2m + 1};$$

or to have the result in whole numbers, we shall get, by taking away the denominators,

$$(4rm^2 + r)^2 - 4r^2m(4m^2 - 1); (4rm^2 + r)^2 \& (4rm^2 + r)^2 + 4r^2m(4m^2 - 1).$$

Now, if r and m be taken each equal to unity, we have 13, 25, and 37 for the three least whole numbers that answer the conditions of the question.

FOURTH SOLUTION, by Mr. MARRAT, Boston.

Let $x^2 - y$, x^2 , and $x^2 + y$ denote the progression; then, by the question $x^2 - 2y$, $x^2 + 2y$, and x^2 are to be square numbers: the last is a square.

Now, suppose $x^2 - 2y = (q - x)^2$, and $x^2 + 2y = (p - x)^2$; then, by the first $y = qx - \frac{1}{2}q^2$, and from the second $y = \frac{1}{2}p^2 - px$; therefore, $qx - \frac{1}{2}q^2 = \frac{1}{2}p^2 - px$, and $x = (p^2 + q^2) \div (2p + 2q)$, where p and q may be taken at pleasure. If $p = 2$, and $q = 1$,

then $x = \frac{5}{6}$ and $y = \frac{1}{2}p^2 - px = \frac{2}{6}$, therefore, $x^2 - y = \frac{13}{36}$,

$x^2 = \frac{25}{36}$ and $x^2 + y = \frac{37}{36}$; and, therefore, rejecting the com-

mon denominators, the numbers are 13, 25, and 37.

FIFTH SOLUTION, by Mr. JOHN CORNTHWAITE.

Put $x^2 + x$, $x^2 + 2x$ and $x^2 + 3x$ for the three numbers; then the difference between the sum of the first and second and the third is equal to x^2 . Hence we have only to make $x^2 + 2x$ and $x^2 + 4x$ squares; the difference of these is $2x$, which being resolved into the factors $2x$ and 1, we may put $(x + \frac{1}{2})^2$, the square of half the sum of the factors equal to the greater of the expressions to be made squares; that is, we may put $(x + \frac{1}{2})^2 =$

$x^2 + 4x$: this equation reduced gives $x = \frac{1}{12}$; and the three

numbers are $\frac{13}{144}$, $\frac{25}{144}$, and $\frac{37}{144}$; or, rejecting the denominators 13, 25, and 37.

Other Solutions were received from Messrs. Boole, Cavill, Codling, Collins, Lowe, Morley, Myers, Pickering, Scurr, and Thompson.

VI. QUESTION 6, by Mr. T. MILNER, Sheffield.

If two rings or circles of equal radii touch each other on their circumferences at P, and one of them be made to revolve round the periphery of the other, so that the area described by the point P may be equal to 2.68 square inches. *Query* the diameter of a circle equal to that of either ring?

FIRST SOLUTION, by SCOTICUS.

Let A (Fig. 5, Pl. 1.) be the centre of the fixed circle, in which P is the point of contact of the two circles at the beginning of the motion; let B be the point of contact when the revolving circle BP'E has any other position, take the arch BP' in the revolving circle equal to the arch BP of the fixed circle, then P' will evidently be a point in the curve, which is the *locus* of the revolving point. Join PP' (which will be parallel to ABC a line joining the centres of the circles) and draw PD, P'D' perpendicular to AC. Put r for the radius of the equal circles, and ϕ for the indefinite angle PAB or FPP' then $BD = BD' = r(1 - \cos. \phi)$ and $PP' = DD' = 2r(1 - \cos. \phi)$, and since the fluxion of the area described by the line PP' revolving about P is equal to $\frac{1}{2}PP'$ multiplied into the fluxion of the circular arch whose radius is PP', that is to

$$2r^2 \dot{\phi} (1 - \cos. \phi)^2 = r^2 \dot{\phi} (3 - 4 \cos. \phi + \cos. 2\phi);$$

by taking the fluent, we have the area itself equal to

$$r^2 (3\phi - 4 \sin. \phi + \frac{1}{2} \sin. 2\phi) :$$

which expression, by taking $\phi = 360^\circ$ gives the whole area = $3 \times 2 \times 3.1416 \times r^2$; but, by the question, the area = 2.68 square inches: hence $2r$, the diameter of either circle is easily found = .7541 the answer.

SECOND SOLUTION, by Mr. JOHN WRIGHT, *Norley.*

Let O (Fig. 6, Pl. 1.) be the centre, and PF a diameter of the circle or ring PIF, about which the equal circle AIH is supposed to roll; with the centre F and radius FP describe a circle cutting PF produced in C, and the circle AIH in H: then, if the chord HA be a tangent to the circle PHC at H, the point A will be the situation of P, when the circles are in contact at I. For, draw the radius OI, which produce till it meet the chord AH in D, and join AI, FH, PI, and HI. Then, since the circles touch each other at I, it is manifest that AD = DH, and because PO = FO, the lines PA, OD, FH are parallel: consequently, the points P, I, H, (PO, PF being respectively equal to OI and FH) are in a straight line: hence PI = IH = AI, and the arch PI = AI, or A is a point in the curve PAC described by the motion of the point P.

Moreover, demit HQ perpendicular to PF and parallel thereto draw the radius FN cutting HB parallel to PF in B. Since FH is parallel to PA, it is well known that the fluxion of the curvilinear segment PAP is

$$\begin{aligned}
 &= \frac{PA \cdot PH}{2FH} = \frac{PQ \cdot PQ}{2HQ} = \frac{(PF \pm \frac{1}{2}FQ) \cdot QF \cdot QF}{HQ} \\
 &\mp \frac{\frac{1}{2}PF \cdot QF}{HQ} = - PF \cdot HQ \pm \frac{1}{2}FQ \cdot HQ \mp \frac{1}{2}PF \cdot HN \\
 &= - FN \cdot HQ \mp \frac{1}{2}HB \cdot BN \mp \frac{1}{2}PF \cdot HN; \text{ the fluent or quadra-} \\
 &\text{ture PAP is, therefore, } FN \cdot HQ \mp \frac{1}{2} \text{ seg. HBN } \mp \text{sector FHN,} \\
 &\text{which being corrected by subtracting therefrom its value at the} \\
 &\text{initial point of the curve, we shall have } - FN \cdot HQ \mp \frac{1}{2} \text{ seg.}
 \end{aligned}$$

HBN $\pm$ sector FHN $+$ $\frac{3}{2}$ quadrant PFN, for the true quadrature PAP; and this, when H coincides with C, becomes

$\frac{3}{2}$ of the semi-circle PNC. Hence the area of the whole curve

PACP, described by a complete revolution of P, is equal to 3 of the semi-circles PNC, and, consequently, the semi-circle

PNC = $\frac{1}{3}$ of the given quadrature; that is = $\frac{2 \cdot 68}{3}$, and its radius

or the required diameter of the ring FP = 7541.

Other Solutions were given by Messrs. Boole, Cavill, Codling, Collins, Milner, Morley, Myers, Pickering, Scurr, and Thompson,

VII. QUESTION 7, by Mr. JOHN WHITLEY, *Altercliffe.*

Let FE, the diameter of a circle described about any plane triangle ACB, bisect the base AB perpendicularly in H, and let EP, FS be drawn perpendicular to AC; with the radii CH, CS and centre C describe the circles CQG and CSD meeting AC in G and S; and let S be the nearest point to C, then from P and G draw the tangents PQ, GD and the rectangle PQ, GD will be equal to the area of the triangle ACB. Required the demonstration.

FIRST SOLUTION, by Mr. THOMAS BAZLEY, *Bolton.*

Fig. 7, Pl. 1, being constructed as per question, it is well known that CS = AP = half the difference, and CP or AS = half the sum of the sides of the triangle.

Now, by well known properties of triangles, we have

$$CH^2 = CQ^2 = \frac{1}{2}(AC^2 + CB^2 - \frac{1}{2}AB^2), \text{ also } CP^2 = \left(\frac{AC+CB}{2}\right)^2;$$

$$\text{whence } CP^2 - CQ^2 = PQ^2 = \left(\frac{AB}{2}\right)^2 - \left(\frac{AC-CB}{2}\right)^2, \text{ also}$$

$$GD^2 = CG^2 - CD^2 = CH^2 - CS^2 = \frac{1}{2}(AC^2 + BC^2 - \frac{1}{2}AB^2) - \left(\frac{AC-CB}{2}\right)^2; \text{ that is, } GD^2 = \left(\frac{AC+CB}{2}\right)^2 - \left(\frac{AB}{2}\right)^2 \text{ as will}$$

easily appear by expanding the expressions. Hence we obtain PQ $\times$ GD a mean proportional between these two quantities $\left(\frac{AB}{2}\right)^2 - \left(\frac{AC-CB}{2}\right)^2$ and $\left(\frac{AC+CB}{2}\right)^2 - \left(\frac{AB}{2}\right)^2$; which is

shewn to be equal to the area of the triangle at Cor. 2. Prop. 3. B. II. of Emerson's Geometry.

SECOND SOLUTION, by Mr. THOS. HOPPER, *Manchester.*

Fig. 8, Pl. 1. The figure being constructed as directed by the question, join CE, and let OM be the radius of the inscribed circle, and I'K the radius of a circle touching the base, and the continuation of the sides, then PK = PM = AH; AP = SC = half the difference of the sides, CP = half the sum of the sides, and CK = half the perimeter. From P through H draw PL meeting CI' in L;

Then

Then $HC^2 - HE^2 = PC^2 - PE^2$ (Simp. Geo. 9. 2.); consequently, $PC^2 - HC^2 = PE^2 - HE^2$:

But $PQ^2 = PC^2 - PH^2$; also if CR be drawn parallel to AB and PR joined, we shall have $PE^2 = REH = RHE + HE^2$; consequently, $PE^2 - HE^2 = RHE$;

and, therefore, $PQ^2 = RHE = ANB = IK \times ON$, a well-known property.

Also $2AH^2 + 2CH^2 = AC^2 + CB^2$ (Simp. Geo. 11. 2.) = $2PC^2 + 2AP^2$ ($2CS^2$);

consequently, $CH^2 - CS^2 = PC^2 - AH^2$: but $GD^2 = CH^2 - CS^2$; therefore, $GD^2 = PC^2 - AH^2 = (PC + AH)(PC - AH) = (PC + PK)(PC - PM) = CK \cdot CM$.

Now, since $PQ^2 = IK \times ON$ and $GD^2 = CK \cdot CM$, it will be as $IK : PQ :: PQ : OM$, and $CM : GD :: GD : CK$;

therefore, $IK \times CM : PQ \times GD :: PQ \times GD : OM \times CK$:

But it is well known that $IK \times CM = OM \times CK$ = the area of the triangle ABC. Consequently, $PQ \times GD$ will also be equal to the area of the triangle ABC.

Q. E. D.

THIRD SOLUTION, by Mr. JOHN BUTTERWORTH, *Haggate, near Oldham.*

The figure being drawn as before. Then $PC^2 - CQ^2$ (CH^2) = PQ^2 ; but, by the theorem on page 39 of the Gentleman's Diary for 1803, $PC^2 - CH^2 = EH \cdot HR$, therof. $PQ^2 = EH \cdot HR$.

Again $CG^2 = (CH^2) - CD^2$ (CS^2) = GD^2 , but $CH^2 = CR^2 + RH^2 = FR \cdot RE + RH^2$,

and $CS^2 = FR \cdot EH$; therof. $GD^2 = FR \cdot RE + RH^2 - FR \cdot EH = FR \cdot RH + RH^2 = FH \cdot RH$; and, consequently,

$PQ^2 \cdot QD^2 = HE \cdot HR \cdot FH \cdot RH = AH^2 \cdot HK^2$, or $PQ \cdot GD = AH \cdot HK$.

Q. E. D.

FOURTH SOLUTION, by Mr. JOHN FAREY, *No. 12, Crown Street, Westminster.*

Fig. 7, Pl. 1. Having constructed the figure as directed in the Question, and also drawn the radii CQ and CD to the points of contact of the tangents PQ and GD; also joined CH, and drawn the perpendicular CI upon the base AB of the triangle; then, because of the right-angled triangles PQC and GDC, we have

have $PC^2 - CQ^2 = PQ^2$, and $CG^2 - CD^2 = GD^2$; and since by the question $PQ \times GD = \frac{1}{2} AB \times CI$, or $PQ^2 \times GD^2 = \frac{1}{4} AB^2 \times CI^2$, therefore, $(PC^2 - CQ^2) \times (CG^2 - CD^2) = \frac{1}{4} AB^2 \times CI^2$. Now, (by *Leybourn's Math. Repository*, Vol. I. p. 284, and Vol. II. p. 101.) PC is equal to half the sum of the sides AC and AB of the triangle ACB, and CS or CD is equal to half the difference of the sides of the same triangle; it being also evident from the construction that CQ and CG are each equal to the line from the vertex bisecting the base of the same triangle, or CH. We may, therefore, infer from the question, the following

THEOREM. *In any plane triangle, the difference between the square of half the sum of the sides and the square of the line bisecting the base, multiplied into the difference between the square of the line bisecting the base and the square of half the difference of the sides, is equal to the square of the area of the triangle.*

Before proceeding to demonstrate this Theorem, I beg permission to mention that it has long appeared to me a desirable thing, to have the several properties of Triangles which the different geometrical and trigonometrical writers have demonstrated, expressed each by an equation, in the characters or symbols adopted by *Lawson* and *Leybourn*, for expressing the parts of a triangle, in their *Synopsis* or *Tables of the Data*, from which triangles have been constructed; and I have been from time to collecting materials, which I hope that leisure will permit me e'er long to arrange for publication, not only on the properties of triangles which can be thus expressed in an equation, but on the *loci* or *known parts* of a triangle, when one, two or three things relating thereto are given. As a specimen of the facility with which certain known properties of a triangle when thus expressed, can be combined, to demonstrate the above, and most other theorems, or to discover new properties; in the above Theorem I substitute (as is done in *Leybourn's Synopsis of Data for the Constructions of Triangles*, page iv.) B for the base, P for the perpendicular from the vertical angle (V) upon the base, S and s for the greater and less sides (subtending V) respectively, m and n for the greater and less segments of the base by the perpendicular from V upon B, and l for the line from the vertex to the middle of the base,

or line bisecting the base; and we have $\left(\left(\frac{S+s}{2}\right)^2 - l^2\right) \times$

$\left(l^2 - \left(\frac{S-s}{2}\right)^2\right) = \frac{1}{4} B^2 \times P^2$. Or, by reducing the fractions

and arranging the terms, to the order of their component parts

B

B, P, S, s, m, n, and l, as they should be for more perfect arrangement in a general *Synoptical Table* of such equations or properties, we have $4B^2 \times P^2 = ((S+s)^2 - 4l^2) \times (-(S-s)^2 + 4l^2)$; and by multiplying the terms of the last part of this equation, we get

$4B^2 \times P^2 = -S^4 + 2S^2 \times s^2 - s^4 + 8S^2 \times l^2 + 8s^2 \times l^2 - 16l^4 = -(S^2 - s^2)^2 + 8l^2 \times (S^2 + s^2 - 2l^2)$. Now, by *Leybourn's Repository*, Vol. I. p. 17, cor. $\frac{1}{2}B^2 = S^2 + s^2 - 2l^2$; therefore, $4B^2 \times P^2 = -(S^2 - s^2)^2 + 8l^2 \times \frac{1}{2}B^2$, and $4B^2 \times l^2 - 4B^2 P^2 = (S^2 - s^2)^2$, or $4B^2 \times (l^2 - P^2) = (S^2 - s^2)^2$. But since

$HC^2 - CI^2 = HI^2$ (see the figure) and $\frac{AI - IB}{2} = HI =$

$\frac{m-n}{2}$, therefore, $l^2 - P^2 = (\frac{m-n}{2})^2$, and we have, $4B^2 \times$

$\frac{1}{4}(m-n)^2 = (S^2 - s^2)^2$; and, lastly, by extracting the root, $B \times (m-n) = S^2 - s^2$, which is demonstrated to be true by *Thomas Simpson* (see his *Geometry Book II. prop. 9, cor. 2.*) and several other writers; our theorem, therefore, and the proposition in the question from whence it is derived are true.

Q. E. D.

Demonstrations were also received from Messrs. Codling, Collins, Scoticus, and Whitley.

VIII. QUESTION 8, by Mr. I. T. M'DONELD.

If on any given right line AB, there be taken any variable distance AL, and from L, in the same direction, any given invariable distance LM; and if with the centres L, B and radii LA, BM arcs be described, it is required to determine the nature of the curve, which is the locus of P the point of their intersection?

SOLUTION by Mr. J. I.

Fig. 9, Pl. 1. Let $AB = a$; $LM = b$; $BM = BP = \rho$; and having drawn PO perpendicular to AB, put $BO = x$. Then $BL = \rho + b$; $LO = \rho + b - x$; $LP = AL = a - b - \rho$; Then, because $LP^2 - LO^2 = BP^2 - BO^2$, we have in symbols $(a - b - \rho)^2 - (\rho + b - x)^2 = \rho^2 - x^2$; whence, $a^2 - 2(a - x)b \pm \rho^2 \pm 2(a - x)\rho$; and adding $-b^2 + b^2 + (a - x)^2$ to one side

side and its equal $(a - x)^2$ to the other side, there results $a^2 - b^2 + (a - b - x)^2 = (a - x + \rho)^2$. Take $AC = LM = b$, draw CD perpendicular to AB , and make $AD = AB = a$; then $DC^2 = a^2 - b^2$; $CO^2 = (a - b - x)^2$ and $(OA + BP)^2 = (a - x + \rho)^2$; whence, we have $DO = AO + PB$ or $BP = DO - AO$.

From what is done it will be easy to derive an algebraic equation for the rectangular co-ordinates of the curve; for we have only to put $PO = y$, to substitute $\sqrt{(x^2 + y^2)}$ for ρ , and to clear the equation of radicals. The equation thus found will shew the curve to be of the fourth order; but the figure of the curve and its principal properties may be more readily deduced from the property investigated above, viz. $PB = DO - AO$. The curve will consist of two equal and similar parts lying on different sides of AB ; it will be a sort of oval including the point B on every side. And if we take $CH = CA$ and produce AB to K making $3BK = BH$, the curve will cut the line AB in H and K .

Solutions were also given by Messrs. Bazley and Caviil.

IX. QUESTION 9, by Mr. JOHN WRIGHT, Norley.

From two given points in the diameter of a given semicircle, it is required, geometrically, to draw two right lines to meet each other at some point in the periphery, so that the sum or difference of the respective rectangles under these two lines and two given ones (M and N) may be equal to a given magnitude.

FIRST SOLUTION, by Mr. WRIGHT, the Proposer.

CONSTRUCTION. Let A and C (fig. 10, pl. 1.) be the two given points in the diameter DH of the given semicircle DBH , of which the centre is O . Produce OD to L till $OL \cdot OA = OD^2$, and make AN perpendicular to LC and equal to a mean proportional between LA and AC ; also from A along AC or AL , according as the sum or difference is given, set off, AE such, that $AE^2 : AN^2 :: M^2 : P \times OC$ (P being a third proportional to AO and N); then through the points N, E, C describe a circle, in which apply the chord NI , such, that $NI \times N$ may be equal to the given sum or difference; and having, on NI dropt the perpendicular CF ,
from

from C to the semicircle DBH apply $CB = FN$; then join AB and the thing is done.

DEMONSTRATION. Join OB, and parallel thereto draw CS meeting AB produced in S; also through the points L, A, B describe a circle cutting CB produced in G, and join AG, CI, NE. Then, since the angles NEA, NIC are equal, the right-angled triangles NEA, FIC are similar, and $IF^2 : FC^2 :: AE^2 : AN^2 :: M^2 : P \times OC$, by construction, that is, $IF^2 : M^2 :: FC^2 : P \times OC$. Because, by construction, $AL \times AO = OD^2$ or OB^2 , OB is a tangent to the circle LABG at B, and, therefore, the angle $AGB = ABO = ASC$; hence the points A, G, S, C are in a circle, and $AB \cdot BS = GB \cdot BC = GC \cdot BC - BC^2$; but by the circle $GC \cdot BC = CL \cdot CA = CN^2$; wherefore, $AB \times BS = CN^2 - CB^2 (FN^2) = FC^2$. Now $AB^2 : AB \cdot BS (FC^2) :: AB : BS ::$ (by parallels) $AO : OC :: AO \times P : OC \times P$, and by alternation, $AB^2 : AO \times P = N^2 :: FC^2 : OC \times P$; hence $AB^2 : N^2 :: IF^2 : M^2$, or $AB : N :: FI (= IN \mp FN) : M$; therefore, $AB \times M = IN \times N \mp FN \times N (= BC \times N)$, and consequently $AB \times M \pm BC \times N = IN \times N$.

Q. E. D.

SECOND SOLUTION, by SCOTICUS.

Let A and B (fig. 11, pl. 1.) be the given points in the diameter, and C the centre of the given circle. Let us suppose the problem resolved, or that AD, DB are drawn to a point D in the circumference, so that $M \times AD \pm N \times DB = M \times L$, a given space. Join CD; then, by a well known proposition in Geometry, (see *Math. Repository*, Vol. I. Page 18, *Old Series*)

$$AD^2 + \frac{CA}{BC} BD^2 = BA \times AC + \frac{BA}{BC} CD^2. \text{ Now BA, AC,}$$

BC, CD are lines given in magnitude, therefore,

$$AD^2 + \frac{CA}{BC} BD^2 = \text{a given space. Let us for the sake of}$$

perspicuity put $AD = x$ and $\frac{\sqrt{(CA \times CB)}}{CA} \times BD = y$, also

$$\frac{N \times \sqrt{(CB \times CA)}}{CA} = P, \text{ where P is evidently a given line,}$$

then, to determine the magnitudes of x and y , we have

$x^2 + y^2 = Q^2$ a given space

$x \pm \frac{P}{M}y = L$ a given line.

And from these equations it is obvious how the values of x and y may be readily determined by various methods strictly geometrical, but as these methods only serve to determine the magnitudes of the lines AD and BD without any regard to their position, it seems unnecessary to point them out as they must be sufficiently obvious to the reader.

THIRD SOLUTION, by Mr. JOHN JOHNSON, *Birmingham.*

Let A, B, (fig. 12, pl. 1.) be the given points in the diameter DE, and AC, BC the lines required to be drawn.

To the centre O draw the radius CO and make $M \times Q =$ to the given space, then $AC \pm BC \times \frac{N}{M} = Q$, and by Prop. 6. of Dr. Stewart's General Theorems,

$AC^2 + BC^2 \times \frac{OA}{OB} = BA \times AO + BA \times CO \times \frac{CO}{BO} =$ to a given space.

Hence the problem is reduced to this; to find two lines AC, BC such, that the sum or difference of AC and a line to which BC has a given ratio may be equal to a given line, and the square of AC together with the space to which the square of BC has a given ratio may be equal to a given space. To effect this, take (in fig. 13, pl. 1.) $Ia = Q$ and $(aR)^2 =$ to the given space make RVT perpendicular to IR, and such that $RV : IR = M : N$ and $RT^2 : RV^2 = OA : OB$; draw To parallel to the line joining IV and apply $ao = aR$; then if the perpendicular oc be demitted cutting IV produced at b; ac, cb are the lines required.

For, by simillar triangles $Ic : cb :: IR : RV :: N : M$, hence, $Ic = \frac{N}{M} \times cb$ and $ac + \frac{N}{M} \times cb = Ia = Q =$ the given line. Again, by parallels $RV^2 : RT^2 :: cb^2 :: co^2 :: oB : oA$, therefore, $co = \frac{OA}{OB} \times cb^2$; consequently $ac^2 + \frac{OA}{OB} \times cb^2 = ao^2 = AR^2 =$ the given space. The letters to

the

the left of oc are applicable to the sum, and those on the right to the difference.

Solutions were also received from Messrs. Butterworth and Smith.

X. QUESTION 10, by Mr. WILLIAM SMITH, Liverpool.

To determine the locus of the vertex of a triangle, the difference of whose sides and the radius of whose inscribed circle are given.

SOLUTION, by Mr. JOHN FAREY, No. 12, Crown Street, Westminster.

Let LKG (fig. 14, pl. 1.) be the given inscribed circle, and the indefinite line AB a tangent thereto at G, let GE be a diameter of the circle, and therefore perpendicular to AB; take GF = the given difference of the sides, then through the points F and E draw the indefinite line FEC, which is the required locus of the vertex.

DEMONSTRATION. From any point C in the line EC, draw tangents to the circle, cutting AB in A and B, and touching the circle in L and K; then, since by the property of tangents $CL = CK$, $AL = AG$ and $BK = BG$, the difference of AL and KB is the same as the difference of the sides AC and CB, equal also to the difference of AG and BG, = FG by construction. Bisect FG in M which will be the middle point of the base; then, because $GE = 2Gc$, and $FG = 2MG$, Mc is parallel to FC, and $FG = AC - CB$.

Q. E. D.

COROLLARY, Hence in any plane triangle, having a circle inscribed therein, and a diameter thereto drawn perpendicular to the base; if a right line be drawn from the vertex of the triangle through the top of the diameter of that circle it will cut off a segment of the base adjacent to the bottom of that diameter, or point of contact with the circle, equal to the difference of the sides of the triangle,

SCHOLIUM. The discovery of the above locus presents an easy method of constructing several cases of triangles, some of which have not yet, to my knowledge, been constructed. In

what follows I shall use the notation and arrangement of *Leybourn's Synopsis of Data for Constructing Triangles*, (see Page iv. of that Work) and earnestly recommend to the correspondents of the *Mathematical Repository*, to adopt the same notation and arrangement of the *Data* in proposing future problems or theorems relating to triangles.

In all the following constructions the given inscribed circle, and a tangent thereto for the base of the triangle is to be first drawn as above, and a diameter EG perpendicular to the base; also the given difference of the sides GF is to be set off on the base and bisected in M for the middle of the base, and the indefinite line FEC drawn for the locus of the vertex, then

1. V. S — s. r. (*See No. 51 in the Synopses.*)

Construct a right-angled triangle, whose angle at the base is equal to $\frac{1}{2}V$ and its perpendicular or opposite side = r , (or if V is given in degrees, take a fourth proportional to *nat. sine* $\frac{1}{2}V$, 1, and r) and with the hypotenuse or this line as radius and centre c (the centre of the given inscribed circle) describe an arc cutting EC in C, draw CA and CB tangents to the inscribed circle cutting AB in A and B, and ABC is the triangle required.

2. B. S — s, r.

From M the middle of B take MB and MA each equal to half B, and from A and B draw tangents to the inscribed circle which will intersect in EC, in the point C (from what is done above) and form the required triangle ABC.

3. — B + S + s. S — s. r.

By the Student, page 34, cor. 2, half the first datum is equal to the segments of S and s next V, by the inscribed circle, = CL or CK; therefore, find the hypotenuse of a right-angled triangle, whose legs are CK and Kc, and therewith proceed, as in the first case.

4. P. S — s. r.

Parallel to AB at the distance of P, draw a line cutting EC in C, from whence draw tangents, as in the 1st case, and ABC is the required triangle.

5. S — s. S. r. (*No. 163 Synopsis.*)

Between the given lines GB and EC, draw a line = S, to touch the given inscribed circle; let it intersect GB in B and EC in C; from C draw a tangent to the inscribed circle, intersecting GA in A, and ABC is the required triangle.

Note. If with S — s there be given $S + s$, $S \times s$, $S : s$; $S^2 + s^2$, $S^2 - s^2$, &c. by known problems or *lineal sections* S and

and s will be known, and the construction as in the last case.

6. $S - s. A$ or $a. r.$

Construct a right-angled triangle, whose angle at the base is equal to $\frac{1}{2}A$ or $\frac{1}{2}a$, and its perpendicular or opposite side $= r$, (or if A or a be given in degrees, take a fourth proportional to *nat. sine* $\frac{1}{2}A$, 1, and r) and with the hypotenuse or this line as a radius and centre c describe an arc cutting GB in B , from B draw a tangent to the inscribed circle cutting EC in C , and from C draw another tangent to the same circle, cutting AB in A , and ABC is the triangle required.

7. $S - s. A - a. r.$ (No. 175 Synopsis.)

From c draw cC making with cE an angle equal $\frac{1}{2}(A - a)$, and intersecting EC in C (See Repository, Vol. I. Page 284, Prop. 2, Old Series), then proceed as in case 1st to complete the triangle ABC .

8. $S - s. m$ or $n. r.$

Suppose the segment AI given, then because $AF = GB$ and FG given, $2GI + IB$ is given, or its half GZ , supposing IB bisected in Z . Draw ZQ perpendicular to AB ; then because it bisects IB it will also bisect BC in Q , and therefore the locus of the point Q will be a straight line passing through the middle of FB and parallel to FC ; consequently the point Q is determined, and the construction obvious.

9. $S - s. m - n. r.$ (No. 177 Synopsis.)

Take $MI = \frac{m-a}{2}$, and erect the perpendicular IC and complete the triangle as in the first case.

10. $S - s. m - n$, segs. by $L. r.$

Take $MD = \frac{m-n}{2}$, through c draw DcC , cutting EC in C and complete the triangle as in the 1st case.

11. $S - s. m \times n$, segs. by $L + L^2. r.$

By Simpson's Geometry III, 26, the second datum is equal to $S \times s$, whence by *Lineal Sections*, the problem is reduced to the 5th case.

12. $S - s. m$ or n , segs. by inscribed circle. $r.$

Take $GB = n$, and proceed as in the 6th case.

13. $S - s. l. r.$

On M as a centre, with the radius l (the line bisecting B) describe an arc cutting EC in C , and proceed as in the 1st case.

14. $S - s$. r . distance of V from centre of the inscribed circle.

This construction is the same as the 1st case.

15. $S - s$. r . Nearest distance of V from the inscribed circle, With radius $r +$ the given distance, and centre c describe an arc cutting EC in C , and proceed as in the 1st case.

16. $S - s$. r . Distance of V from contact of B and inscribed circle.

With the given distance as radius and G as centre describe an arc cutting EC in C as before.

17. $S - s$. r . Distance of A or a from centre of inscribed circle.

This is to be done as the 6th case.

18. $S - s$. r . Nearest distance of A or a to inscribed circle. Proceed as in the 17th and 6th cases.

19. $S - s$. r . Distance of centre of inscribed circle from intersection of B and L .

With the given distance as radius and centre c , describe an arc cutting AB in D , and proceed as in the 10th case.

20. $S - s$. r . Distance of centre of inscribed circle from foot of P .

With the given distance as radius and centre c describe an arc cutting AB in I , erect the perpendicular IC meeting the locus in C , and proceed as in the 1st case.

21. $S - s$. r . Distance of centres of inscribed and circumscribed circles.

The diameter of the circumscribing circle becomes known from Prop. V, Art. II. Vol. II. of the Old Series, and the construction from thence obvious.

The known *loci*, when the following pairs of data are given, viz. V and B , V and $B + S + s$, V and $-B + S + s$, V and $\text{infc. } \square$ on B , B and $S : s$, B and $S^2 + s^2$, B and $S^2 - s^2$, $S : n$ and L , $A - a$ and $m - n$, and others, present each a similar series of constructions, obvious to the geometer almost at first sight; many of which, from the *limits* of one of the data being shewn, offer a much greater number of cases than are given above, I beg leave to recommend to your ingenious correspondents, a search to discover as many of these *loci* as possible, even though they happen to be conic sections, or even curves of higher orders; these investigations appearing to offer one

one of the most pleasing, and certainly not the least useful applications of Algebra to Geometry.

The Solutions given by Messrs. Bazley, Butterworth, Dawes, Smith, and Wright, were exactly the same as that given above by Mr. Farey.

XI. QUESTION 11, by Mr. JOHN WRIGHT, Norley.

Given the base and vertical angle of a plane triangle to construct it, when the solid of the line bisecting the vertical angle into the square of the difference of the segments of the base made by the perpendicular is a maximum.

FIRST SOLUTION, by Mr. JOHN WRIGHT, the Proposer.

CONSTRUCTION. Through the extremities B, H, (fig. 15, pl. 1.) of the given base describe a circle BCHP such, that the segment ACH may contain the given vertical angle; bisect BH in A with the diameter CP, in which determine the point S such that, R being the centre of the circle, and $RI = CI$, the rectangle $SP \times SI$ may be equal to $AP \times RI$: draw SQ parallel to BH meeting the circle in Q, then joining BQ, HQ, BQH will be the triangle required.

DEMONSTRATION. Demit QD perpendicular to BH and DG to PQ; to the circle at Q draw the tangent TN terminating in BH and PC produced, and let TP be bisected in O. Then because $TS : PS :: CS : RS$, by composition $TP = 2PO : PS :: CR = 2RI : RS$, or $PO : PO + OS :: RI : RS$, and by div. $PO : OS :: RI : IS :: SP : AP$ by constr. therefore, $OS \times PS = PO \times AP$; hence by my solution of Question 298, No. XII. of the Old Series of the Repository, the right-angled triangle FQD is greater than any other whose hypotenuse passes through P, and having one leg parallel to TP, that can be formed within the triangle ATN, and much more then must it be greater than any other of a like construction that can be formed within the segment HQCB. The triangle FQD being then a maximum, $FQ \times GD$, and consequently $FQ \times GD \times CP$ must be a maximum; but $GD \times CP = FQ \times SQ$ by similar triangles; consequently $FQ^2 \times SQ$ or $FQ^2 \times (BD - DH)$ is a maximum.

Q. E. D.

SECOND

SECOND SOLUTION, *by Mr. WILLIAM SMITH, Liverpool.*

CONSTRUCTION. Let AB (fig. 16, pl. 2.) be the given base, AGBF the given circumscribing circle, and GLMF its diameter, bisecting the base in M. On GM describe a semi-circle, to which draw a tangent PQN the method of doing which is well known) meeting FG produced in Q and the tangent at F in P, so that QL may be equal to 4LF, draw LNC parallel to AB meeting the circle in C; join AC, CB and ACB will be the triangle required.

DEMONSTRATION. Join FC meeting AB in E, and demit the perpendicular CD. Then, by similar triangles $EC \times CL^2 = FC \cdot CL \cdot ED$, but $CL \cdot ED = CD \cdot GL = ML \cdot GL = LN^2$; therefore, $EC \cdot CL^2 = FC \cdot LN^2$; but, since by construction, $QL = 4LF$, the product $FL \cdot LN^4$ is a maximum, or the same multiplied into the constant quantity TG will be a maximum, that is $FC^2 \cdot LN^4$ is a maximum, or $FC \cdot LN^2$ is a maximum, that is, from what is done above, $EC \cdot CL^2$ is a maximum.

Q. E. D.

Ingenious Solutions were also received from Messrs. Bazley, Cavill, and Marrat.

XII. QUESTION 12, *by Mr. JAMES CUNLIFFE, Bolton.*

It is required to determine the *locus* of a point, from whence if perpendiculars be drawn to three straight lines given by position, the sum of the squares of the said perpendiculars may be equal to a given space.

FIRST SOLUTION, *by Mr. J. I.*

1. In the first place we shall consider the proposition in the case when the lines given by position meet in one point.

ANALYSIS. Let AB, AC, AD (fig. 17, pl. 1.) be three lines given by position intersecting one another in the point A, and let EB, EC, ED be the perpendiculars drawn to them from the

the point E. Join AE, and upon it as a diameter describe a circle, which will pass through D, C, B as is manifest. Join DC, CB, DB; then it is obvious that the several angles of the triangle DCB are given (being equal to the angles that the given lines make with one another) and therefore the triangle is given in species. Take any line pr (fig. 18, pl. 1.) and upon it construct a triangle pqr similar to DCB; about the triangle pqr describe a circle of which m is the centre, and take n the centre of gravity of the three points pqr and join pm, qm, rm; pn, qn, rn and draw the diameter umns; also let O be the centre of the circle passing through D, C, B, and G the centre of gravity of the same points, and join DO, OB, OC; DG, BG, GC and draw the radius OGF and the lines AF and EF. Then because the triangle pqr is similar to the triangle DCB and that the points m and n are similarly situated in respect of the one as O and G are in respect of the other, it may easily be proved that the several triangles pmn, qmn, rmn are similar to the triangles DOG, BOG, COG; therefore, the several angles at O are equal to the several angles at m each to each; but the angles DAF, CAF, BAF are the halves of the angles at O; therefore, they are likewise the halves of the angles at m; consequently the angles that AF makes with the given lines AB, AC, AD are given and the line AF is given by position.

Draw GH at right angles to OF and join OH, HF. Because G is the centre of gravity of the three points D, C, B, it is a property very anciently known (Vide Cor. 1st. Prop 5th. Lib. 2d. Loc. Plan. a R. Simfon refl.) that $DE^2 + EC^2 + EB^2 = DG^2 + BG^2 + CG^2 + 3GE^2$: And in like manner, $DO^2 + BO^2 + OC^2$, or $3OH^2 = DG^2 + BG^2 + CG^2 + 3GO^2$; therefore, $DG^2 + BG^2 + CG^2 = 3OH^2 - 3GO^2 = 3GH^2$, consequently denoting by S the given space that is equal to $DE^2 + EB^2 + EC^2$, there results $S = 3GH^2 + 3GE^2 = 3GH^2$

$+ 3GF^2 + 3FE^2 = 3HF^2 + 3FE^2$: therefore, $\frac{S}{3} = HF^2 +$

FE^2 . Draw tn at right angles to ms and join mt, ts: then it is obvious that the triangles mtn, mts are similar to HOG, HOF; therefore, $2tm$ or $su : ts :: 2OD$ or $AE : AF$. Therefore,

$\frac{S}{3} = \frac{ts^2}{su^2} \times AE^2 + EF^2$. Make $ts^2 : su^2 :: \frac{S}{3} : AK^2$, and

$su : ts :: AK : AL$: then it will evidently follow, that,

$$AK^2 = AE^2 + \frac{AL^2}{AK^2} \times EF^2.$$

a property of an ellipse of which A is the centre, AK the semi-transverse axis, and AL the focal distance. Therefore the *Locus* required is a given ellipse.

CONSTRUCTION. Assume any line as pr ; make the angle $qpr = CAB$ and the angle $qrp = CAD$: about the triangle pqr describe a circle of which m is the centre, and take n the centre of gravity of the points qpr , and draw the several lines as directed in the Analysis. Draw the line AF to make the angles DAF , CAF , BAF equal to the halves of the angles tms , qms , rms : and make

$$ts^2 : su^2 :: \frac{S}{3} : AK^2$$

$$\text{and } su : ts :: AK : AL$$

then the locus required will be an ellipse of which A is the centre, AK the semi-transverse axis, and AL the focal distance.

2. We proceed to the case when the given lines do not intersect in one point.

ANALYSIS. Let the given lines form the triangle ABC (fig. 19, pl. 1.) by their mutual intersections; assume a point P within the triangle, and draw PQ , PR , PS perpendicular and PL , PK , PM parallel to AC , AB , and BC ; then, if the perpendiculars drawn from the point E meet PL , PK , and PM in L , K , and M , it readily follows that, $EH^2 + EF^2 + EG^2$

$$= \{EL^2 + EM^2 + EK^2\} + \{PR^2 + PQ^2 + PS^2\} + 2\{EK \cdot PR + EL \cdot PQ - EM \cdot PS\}.$$

Again, twice the area of the triangle ABC is equal to $\{AC \cdot PR + BC \cdot PS + AB \cdot PQ\}$ and it is also equal to $\{AC \cdot EG + AB \cdot EF - BC \cdot EH\}$: therefore, taking equals

from equals, $\{BC \cdot EK + AB \cdot EL - AC \cdot EM\} = 0$. Now,

if we suppose the point P , which was assumed *ad libitum*, to be taken such, that PR , PQ , PS are proportional to the given lines AC , BC , and AB , it is clear that,

$$\{PR \cdot EK + EL \cdot PQ - EM \cdot PS\} = 0.$$

Supposing then that the point P is taken as above, it will follow from what has already been shewn, that $EH^2 + EF^2 + EG^2 =$

$$\{EM^2 + EL^2 + EK^2\} + \{PR^2 + PQ^2 + PS^2\}.$$

Therefore, denoting by S the given space that is equal to $EH^2 + EF^2$

+

+ EG^2 , there results $EL^2 + EM^2 + EK^2 = S - \{PR^2 + PQ^2 + PS^2\}$. Therefore, supposing the point to be determined, this second is manifestly reduced to the first case.

It remains only to determine the point P. Draw the lines APN, CPO, and BP. Then, as was supposed, $AB : AC :: PQ : PR$; therefore, $AB^2 : AC^2 :: AB \times PQ : AC \times PR :: \triangle APB : \triangle APC$. But it is easily shewn that $\triangle APB : \triangle APC :: \triangle ABN : \triangle ACN :: BN : NC$. Therefore, $AB^2 : AC^2 :: BN : NC$; therefore, the point N in which BC is cut in a given ratio is given, and the line AN is given by position. In like manner it is shewn that $BC^2 : CA^2 :: BO : AO$, and that CO is given by position; therefore the point P is determined.

By the preceding Analysis, the second case is reduced to the first. It follows too that the sum of the squares of the perpendiculars drawn from the point is a minimum, or, is less than the sum of the squares of perpendiculars drawn from any other point; and hence, in this second case, the locus is restricted, it being requisite that the given space shall not be greater than $\{PR^2 + PQ^2 + PS^2\}$.

COROLLARY. Although the case of three lines only has been considered, yet it is manifest that the same Analysis and Construction will apply, *mutatis mutandis*, whatever be the number of lines given by position.

SECOND SOLUTION, by Mr. WILLIAM SMITH, *Liverpool*.

LEMMA. Let Y, W, t, &c. (fig. 20, pl. 1.) be any number of given points in a right line given in position, and let any line GP be drawn parallel to a line given in position and produced to P, so that a square to which YG^2 has a given ratio, together with a square to which WG^2 has another given ratio, together with a square to which tG^2 has another given ratio, together with GP^2 may be equal to a given space. Then I say the locus of the point P is a given ellipse.

DEMONSTRATION. Take any point G and erect the perpendicular GE, such that the triangle tGE may be equal to a square to which tG^2 has a given ratio, and produce tE to meet YC, parallel to GE, in C; find the point L so that the triangle YGL may be equal to the square to which YG^2 has the given ratio;

(D 2)

ratio; find also the point R such, that the triangle WGR may be equal to a square to which WG^2 has the given ratio. Then, by hypothesis, the $\triangle tGE + \triangle YGL + \triangle WGR + GP^2 =$ a given space. Produce YL to meet tE in A, a given point, draw AD parallel to YC and join CD meeting GE in S; also produce WR to meet CD in B, a given point, and demit the perpendicular BF. Then the $\triangle tGE + \triangle YGL = \triangle tAY + \triangle AEL$, but YC and AD being parallel, it is evident that $GS = EL$, or the $\triangle AEL =$ the $\triangle DGS$, also that the $\triangle DGS + \triangle WGR = \triangle WBD + \triangle SBR$; draw WZ parallel to FB and join ZF meeting DL in O, then the $\triangle FGL = \triangle SBR$; therefore, the given sum ($= \triangle tGE + \triangle YGL + \triangle WGR + GP^2$) $= \triangle YtA + \triangle WBD + \triangle FGL + GP^2$, but the triangles YtA and WBD are given; therefore, the $\triangle FGL + GP^2$ is given, that is, a square to which FG^2 has a given ratio, together with GP^2 is given, and F a given point; therefore, by the well-known properties of the ellipse, the locus of P is a given ellipse whose centre is F and ordinate GP, parallel to the conjugate of FG.

ANALYSIS. Fig. 21, Pl. 1. Let AB, BC, CD, DA, &c. be any number of lines given in position, and P a point such, that demitting the perpendiculars Pa, Pb, Pc, Pd, the sum of their squares may be equal to a given space. Produce one of the perpendiculars as Pa to meet AD in K, CD in I, and AB in H; at P erect an indefinite perpendicular Pirqpgfv, and take therein Pi such, that $HP \cdot Pi = Pb^2$, and produce Hi to meet aL (drawn at half a right angle to aP) in R, demit the perpendicular RE, also draw HL parallel to RE meeting AR in L, and join EL meeting Pi in f and demit the perpendicular EO upon AB. Then, since $HP \cdot Pi = Pb^2$, by parallels, $HE \cdot Ea (ER) = EO^2$; therefore, the locus of the point E is a right line BE given in position, which produce to meet DC in the given point X. Moreover, since the $\triangle HPi =$ half the square on Pb, and the $\triangle aPv =$ half the square on aP; therefore, half the sum of the squares of Pb, Pa, is equal to half the sum of the triangles HPi, aPv $= \triangle HRa + \triangle Riv$, but since HL is parallel to RE the $\triangle Riv = \triangle EPf$; therefore, half the square on aP together with half the square on bP $= \triangle HaR + \triangle EPf$. Again, find in Pi the point r, such that $IP \cdot Pr$ may be $=$ to the square of the perpendicular Pc demitted on IDC and join Ir meeting EL in S, demit the perpendicular SF, and draw IM parallel thereto, meeting EL in M, and join FM meeting Pr in g, and demit on CD the perpendicular Fv. Then since $IP \cdot Pr = Pc^2$; therefore $IF \cdot FS = Fy^2$; therefore the locus of the point F is a right line XF given in position, which produce to meet AD in Y, a given point; also, since the rect. $IPr = Pc^2$; therefore, half the sum of the squares

squares of Pa , Pb , $Pc = \Delta HaR + \Delta EPf + \Delta IPr = \Delta HaR + \Delta IES + \Delta Srf$, but since FS is parallel to IM , the $\Delta Srf = \Delta FPg$; therefore, half the sum of the squares of Pa , Pb , $Pc = \Delta HaR + \Delta IES + \Delta FPg$. Find also, in Pr the point p , such, that $KP \cdot Pp$ may be $=$ to Pd^2 , join Kp and produce it to meet FM in T , demit the perpendicular TG and draw KN parallel thereto, meeting FM in N , and join GN meeting Pp in q and demit the perpendicular Gz on AD . Then since $KP \cdot Pp = Pd^2$; therefore, $KG \cdot GT = Gz^2$; therefore the locus of the point G is a right line YG , given in position. Moreover, since the $\Delta Kpp =$ half the square on Pd ; therefore, half the sum of the squares of Pa , Pb , Pc , $Pd = \Delta HaR + \Delta IES + \Delta KET + \Delta GPq$, the sum of these triangles is, therefore, equal to a given space, but ΔHaR is to BE^2 in a given ratio, that is half the rectangle $HaraE$ is to BE^2 in a given ratio, the ΔIES is to EX^2 in a given ratio, and the ΔKFT is to YG^2 in a given ratio; also drawing Bt and XW parallel to aG meeting YG in t and W ; BE^2 is to Gt^2 in a given ratio, and EX^2 is to GW^2 in a given ratio; therefore, (the $\Delta HaR + \Delta IES + \Delta KFT + \Delta GPq =$) a square to which Gt^2 has a given ratio, together with a square to which GW^2 has a given ratio, together with a square to which YG^2 has a given ratio, together with a square to which GP^2 has a given ratio, is given by the question; and therefore, by the Lemma, the locus of the point P is a given ellipse.

XIII. QUESTION 13, by SCOTICUS.

Let ϕ and ψ be two arches such that $\sin. \phi + \cos. \psi = a$, and $\sin. \psi + \cos. \phi = b$: Required $\sin. \phi$ and $\cos. \phi$, also $\sin. \psi$ and $\cos. \psi$?

FIRST SOLUTION, by the Proposer.

Put $\theta = 90^\circ - \psi$, then $\cos. \psi = \sin. \theta$ and $\sin. \psi = \cos. \theta$, thus the proposed equations become

$$\sin. \phi + \sin. \theta = a$$

$$\cos. \phi + \cos. \theta = b.$$

Now, from the common theorems for finding the sine and cosine of the sum and difference of two arches, (see Emerson's Trig. Book I. Prop. 6. Schol. 2.) we have

$\sin.$

$$\sin. \phi + \sin. \theta = 2 \sin. \frac{1}{2}(\phi + \theta) \cos. \frac{1}{2}(\phi - \theta)$$

$$\cos. \phi + \cos. \theta = 2 \cos. \frac{1}{2}(\phi + \theta) \cos. \frac{1}{2}(\phi - \theta)$$

therefore,

$$2 \sin. \frac{1}{2}(\phi + \theta) \cos. \frac{1}{2}(\phi - \theta) = a$$

$$2 \cos. \frac{1}{2}(\phi + \theta) \cos. \frac{1}{2}(\phi - \theta) = b.$$

Taking now the sum of the squares of both equations, and observing that $\cos.^2 \frac{1}{2}(\phi + \theta) + \sin.^2 \frac{1}{2}(\phi + \theta) = 1$ we find

$$4 \cos.^2 \frac{1}{2}(\phi - \theta) = a^2 + b^2.$$

Hence we have immediately

$$2 \sin. \frac{1}{2}(\phi - \theta) = \sqrt{4 - a^2 - b^2},$$

and from the same two equations

$$\cos. \frac{1}{2}(\phi + \theta) = \frac{b}{\sqrt{a^2 + b^2}}, \quad \sin. \frac{1}{2}(\phi + \theta) = \frac{a}{\sqrt{a^2 + b^2}}.$$

But by the Prop. in Emerson's Trigonometry already quoted,

$$\sin. \phi - \sin. \theta = 2 \cos. \frac{1}{2}(\phi + \theta) \sin. \frac{1}{2}(\phi - \theta)$$

$$\cos. \phi - \cos. \theta = -2 \sin. \frac{1}{2}(\phi + \theta) \sin. \frac{1}{2}(\phi - \theta).$$

Therefore,

$$\sin. \phi - \sin. \theta = \frac{b\sqrt{4 - a^2 - b^2}}{\sqrt{a^2 + b^2}}$$

$$\cos. \phi - \cos. \theta = \frac{a\sqrt{4 - a^2 - b^2}}{\sqrt{a^2 + b^2}}.$$

Hence, and from the given equations, we derive the following expressions, in which $x = \frac{\sqrt{4 - a^2 - b^2}}{\sqrt{a^2 + b^2}} = \sqrt{\frac{4}{a^2 + b^2} - 1}$.

$$\sin. \phi = \frac{1}{2}(a + bn)$$

$$\cos. \phi = \frac{1}{2}(b - an)$$

$$\cos. \theta = \sin. \psi = \frac{1}{2}(b + an)$$

$$\sin. \theta = \cos. \psi = \frac{1}{2}(a - bn).$$

SECOND SOLUTION, by the Rev. Mr. SCURR.

Let the sines of ϕ and ψ be x and y respectively; then will the cosines be denoted by $\sqrt{1 - x^2}$ and $\sqrt{1 - y^2}$.

By the question $x + \sqrt{1 - y^2} = a,$

and $y + \sqrt{1 - x^2} = b:$

From the first equation $x = a - \sqrt{1 - y^2},$

and

and from the second $x = \sqrt{1 - (b + y)^2}$.

Hence, $a - \sqrt{1 - y^2} = \sqrt{1 - (b + y)^2}$:

This equation by proper reduction becomes,

$$y^2 - by = \frac{a^2}{s^2} - \frac{s^2}{4}, \text{ where } s^2 = a^2 + b^2 :$$

$$\text{Whence } y = \frac{1}{2}b - \sqrt{\frac{4a^2 - s^4 \times b^2 s^2}{4s^2}} = \frac{1}{2}b - \frac{1}{2}a\sqrt{\left(\frac{4}{a^2 + b^2} - 1\right)}.$$

Every thing required by the question may now be very easily found.

THIRD SOLUTION, by Mr. WM. SMITH, *Liverpool*.

LEMMA. Let ZN, NW, WZ (fig. 22, pl. 1.) be three right lines given in position, and let O be a given point ; it is required to find the point S, in WZ such that drawing SE, SC parallel to lines given in position, and meeting ZN, NW in E and C, the lines drawn from O to the points E and C may be equal to each other, that is, so that OE may be equal OC.

Demit the perpendiculars OF, OD, and draw FP, DL parallel to SE, SC ; then, since OE = OC it is plain that DC² - EF² (= OF² - OD²) is given, but DC is to LS, and EF to PS in given ratios ; therefore, PL being a given line, the problem is reduced to the finding the point S therein, such that a square to which SL² has a given ratio, minus a square to which SP² has another given ratio, may be equal to a given space, the method of doing which is very simple. Hence the following construction of the problem proposed by Scoticus.

Take (fig. 23, pl. 1.) OG, OM equal to a and b respectively, and bisect them in I and L, erect the indefinite perpendicular OW, and take OH, OQ equal to OI, OL respectively ; draw QF, HT parallel to OM meeting IF and LT, drawn parallel to OW, in F and T. Take HW so that HW may be to HT as MO to OG, also take QZ such that QZ may be to QF as GO to OM ; join WT, ZF and produce them to meet OM in K and N. Then, by the *Lemma*, find the point S in GV, drawn at half a right angle to GO, such that drawing SC parallel to OM, meeting WK in C, and SE parallel to WO meeting ZN in E, the lines drawn from O to E and C may be equal ; join OE, OC and from the centre O with radius OE or OC describe a circle meeting OM in A, so shall AC, AE be the arches required.

DEMONSTRATION. Demit the perpendiculars ED, CB meeting QF, HT in R and P. Then, since, by construction,

OGS

OGS is half a right angle, $GD (= DS) = CB$, and, by construction, $BP (OH) = GI (OI)$, therefore, $ID = PC$: but $PC : BL (PT)$ (as $HW : HT$) $:: MO : OG$, that is, $ID : BL :: MO : OG$; therefore, $ID \times 2OG = BL \times 2MO$, that is, $GD^2 - DO^2 = MB^2 - OB^2$: but, by the circle $OD^2 + DE^2 = OB^2 + BC^2 (DG^2)$, therefore, $(BC^2) GD^2 - DO^2 = DE^2 - OB^2$, and, therefore, $MB^2 - BO^2 = DE^2 - BO^2$, or $MB = DE$. But, by construction, $MB + BO (= DE + BO) = b$, and $GD + DO = CB + DO = a$; therefore, $AC (\varphi)$ and $AE (\psi)$ are two arches, such that $\sin. \varphi + \cos. \psi = a$, and $\sin. \psi + \cos. \varphi = b$. Consequently AC, AE are the two arches required.

Solutions were also received from Messrs. Dawes and Marrat.

XIV. QUESTION 14, by HYPATIA.

Required the value of $(1 + \frac{1}{n})^n$ when n is indefinitely great.

FIRST SOLUTION, by Mr. THOMAS BAZLEY, Bolton.

Let the given expression $(1 + \frac{1}{n})^n$ be actually involved by the Binomial Theorem, and we obtain

$$1 + n \cdot \frac{1}{n} + n \cdot \frac{n-1}{1} \cdot \frac{1}{n^2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{1}{n^3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{1}{n^4} + \&c.$$

but since n is indefinitely great, the factors $n, n-1, n-2, \&c.$ are equal to each other and to n , hence it becomes

$$1 + \frac{n}{n} + \frac{n^2}{2n^2} + \frac{n^3}{2 \cdot 3n^3} + \frac{n^4}{2 \cdot 3 \cdot 4n^4} + \&c. = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \&c;$$

which series (it is well known) expresses the number 2.718282, whose hyp. log. is unity.

SECOND SOLUTION, by SCOTICUS.

Let $x = (1 + \frac{1}{n})^n$, then taking the hyp. log. of both sides of the equation

log.

$$\log. x = n \times \log. \left(1 + \frac{1}{n}\right)$$

$$\text{but } \log. \left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \&c.$$

$$\text{therefore, } \log. x = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{4n^3} + \&c.$$

and, when n is indefinitely great, $\log. x = 1$. Hence x = the number whose hyp. log. is 1 = 2.7182818, the answer.

THIRD SOLUTION, *by Mr. W. WALLACE, R. M. College.*

Let $v = \left(1 + \frac{x}{n}\right)^n$, then taking the fluxions,

$$\dot{v} = \dot{x} \left(1 + \frac{x}{n}\right)^{n-1};$$

let the latter equation be divided by the former, and we have

$$\frac{\dot{v}}{v} = \frac{\dot{x}}{1 + \frac{x}{n}}$$

which equation, when n is indefinitely great becomes

$$\frac{\dot{v}}{v} = \dot{x}$$

Hence, taking the fluents, $v = e^x$, where $e = 2.7182818$ = the number whose hyp. log. is unity, and when $x=1$, then $v=e$.

XV. QUESTION 15, *by ANALYTICUS.*

Required a general expression for the area of a curve whose equation is

$$y^3(1 - 3x^2) = \frac{a}{3 - x^2}.$$

SOLUTION, *by ANALYTICUS, the Proposer.*

From the given equation we have the fluxion of the area

$$= \frac{a}{3 - x^2} \times \frac{\dot{x}}{\sqrt[3]{(1 - 3x^2)}}.$$

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Assume

Assume $\sqrt[3]{(1 - 3x^3)} = 1 - \frac{x}{z}$, hence, $x = \frac{3z}{2} (1 + z^2) - \frac{3z}{2} \sqrt{(z^4 + 2z^2 - 1)}$, so that by taking the fluxion and substituting, &c. the fluxion of the area is transformed to

$$\frac{\frac{3}{2}a(1 - z^2)\dot{z}}{(1 + z^2)(3z^2 - 1)} - \frac{a}{2\sqrt{3}} \cdot \frac{3z^4 + 1}{3z^4 + 2z^2 - 1} \cdot \frac{\dot{z}}{\sqrt{(3z^4 + 6z^2 - 1)}}$$

And in this expression the first part of the fluxion is rational.

To transform the second part, let us further assume $\sqrt{(3z^4 + 6z^2 - 1)} = vz$, hence the second term becomes

$-\frac{a}{2\sqrt{3}} \cdot \frac{\dot{v}}{v^2 - 4}$, thus upon the whole the fluxion of the area is transformed to

$$\frac{\frac{3}{2}a(1 - z^2)\dot{z}}{(1 + z^2)(3z^2 - 1)} - \frac{a}{2\sqrt{3}} \cdot \frac{\dot{v}}{v^2 - 4}$$

an expression entirely without radicals, and therefore integrable by methods sufficiently known.

XVI. QUESTION 16, by AMICUS.

It is required to find the surface of an oblique cone the radius of the base being unity, the perpendicular from the vertex $\frac{1}{2}$ and the distance of that perpendicular from the centre of the base $\frac{1}{3}\sqrt{3}$.

FIRST SOLUTION, by X.

Let C (fig. 24, pl. 1.) be the centre of the base, V the vertex, VA the perpendicular from the vertex, meeting the base in A: join CA meeting the circumference of the base in H, take B any point in the circumference, and draw the tangent BD, meeting CA produced in E, draw VD perpendicular to BE, and join CB, BA, DA, VE.

The triangles VDE, VAE, VAD are right-angled at D and A, therefore, $VE^2 - EA^2 = VD^2 - DA^2$, hence $AE^2 - AD^2 = VE^2 - VD^2 = DE^2$, therefore, AD is perpendicular to EB, and consequently parallel to CB.

By hypothesis CB, the radius of the base, = 1.

Put

Put CA, the dist. of the perp. from the centre $(= \frac{1}{2}\sqrt{3}) = d$,

VA the perp. from the vertex $(= \frac{1}{2}) = p$,

HB, any variable arch of the circumference $= \phi$.

The triangles ECB, EAD are similar, hence

$$EC : CB :: EA : AD,$$

$$\text{or } \sec. \phi : 1 :: \sec. \phi - d : AD = \frac{\sec. \phi - d}{\sec. \phi} = 1 - d \cos. \phi$$

Therefore, $DV = \sqrt{p^2 + (1 - d \cos. \phi)^2}$; now the fluxion of the surface, comprehended by the arch ϕ , and lines drawn from the vertex to the extremities of that arch, is equal to $\frac{1}{2}\dot{\phi} \times VD$, therefore, putting s for that surface,

$$\dot{s} = \frac{1}{2}\dot{\phi} \sqrt{p^2 + (1 - d \cos. \phi)^2}.$$

To prepare this fluxion for integration let us assume another arch ψ , so related to ϕ that

$$\cos. \phi = \frac{m + \cos. \psi}{1 + m \cos. \psi}, \text{ or } \tan. \frac{1}{2}\phi = \sqrt{\frac{1-m}{1+m}} \tan. \frac{1}{2}\psi,$$

where it appears that while ϕ increases from 0 to π (or 180°) ψ also increases from 0 to π .

From the assumed equation we find

$$\dot{\phi} = \frac{1}{\sin. \phi} \cdot \frac{(1-m^2)\dot{\psi} \sin. \psi}{(1+m \cos. \psi)^2}, \text{ and } \sin. \phi = \frac{\sqrt{(1-m^2)} \sin. \psi}{1+m \cos. \psi},$$

$$\text{and, therefore, } \dot{\phi} = \sqrt{(1-m^2)} \cdot \frac{\dot{\psi}}{1+m \cos. \psi};$$

we next substitute $\frac{m + \cos. \psi}{1 + m \cos. \psi}$ for $\cos. \phi$ in the radical

quantity $\sqrt{p^2 + (1 - d \cos. \phi)^2}$ which thus becomes

$$\frac{\sqrt{a + \beta \cos. \psi + \gamma \cos.^2 \psi}}{1 + m \cos. \psi} \text{ where}$$

$$a = p^2 + (1 - md)^2$$

$$\beta = 2 \{ p^2 m + (m - d)(1 - md) \},$$

$$\gamma = m^2 p^2 + (d - m)^2.$$

Assume $\beta = 0$; then, by resolving a quadratic equation, we

$$\text{find } m = \frac{1 + p^2 + d^2}{2d} - \sqrt{\left\{ \left(\frac{1 + p^2 + d^2}{2d} \right)^2 - 1 \right\}}.$$

where the least root is taken, in order that m may be less than unity, and thus ψ a real quantity.

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The

The radical $\sqrt{\alpha + \beta \text{ cof. } \psi + \gamma \text{ cof.}^2 \psi}$ may now be expressed thus $\sqrt{\alpha + \gamma} \times \sqrt{1 - \frac{\gamma}{\alpha + \gamma} \text{ fin.}^2 \psi}$: now from the preceding equations we have

$$\alpha + \gamma = (1 + m^2) p^2 + (1 - md)^2 + (d - m)^2$$

$$mp^2 = -(m - d)(1 - md)$$

hence, by exterminating p^2 we have more simply

$$\alpha + \gamma = (1 - m^2)^2 \frac{d}{m}.$$

Again, from these equations

$$\alpha = p^2 + (1 - md)^2$$

$$m^2 p^2 = (d - m)^2 (1 - md)^2$$

$$\gamma = m^2 p^2 + (d - m)^2$$

by exterminating $m^2 p^2$ and $1 - md$ we find

$$\alpha (d - m)^2 = \gamma p^2, \text{ and hence } \frac{\gamma}{\alpha + \gamma} = \frac{(d - m)^2}{p^2 + (d - m)^2},$$

let this value of $\frac{\gamma}{\alpha + \gamma}$ be denoted by e^2 , and our original

$$\text{fluxionary equation } \dot{s} = \frac{1}{2} \dot{\phi} \sqrt{p^2 + (1 - d \text{ cof. } \phi)^2}$$

is transformed to

$$\dot{s} = \frac{1}{2} (1 - m^2)^{\frac{3}{2}} \sqrt{\left(\frac{d}{m}\right)} \frac{\dot{\psi} \sqrt{(1 - e^2 \text{ fin.}^2 \psi)}}{(1 + m \text{ cof. } \psi)^2}.$$

We are next to reduce this new fluxionary expression to others more simple, whose fluents are commonly known. For this purpose we assume the algebraic expression

$$\frac{\text{fin. } \psi \sqrt{(1 - e^2 \text{ fin.}^2 \psi)}}{1 + m \text{ cof. } \psi}, \text{ the fluxion of which is}$$

$$\frac{\dot{\psi} \text{ cof. } \psi \Delta}{1 + m \text{ cof. } \psi} - \frac{e^2 \dot{\psi} \text{ fin.}^2 \psi \text{ cof. } \psi}{(1 + m \text{ cof. } \psi) \Delta} + \frac{m \dot{\psi} \text{ fin.}^2 \psi \Delta}{(1 + m \text{ cof. } \psi)^2}$$

where, for the sake of brevity, Δ is put for the radical $\sqrt{(1 - e^2 \text{ fin.}^2 \psi)}$; and here it is evident that this fluxionary expression, the fluent of which we already know, involves in its terms the

fluxion $\frac{\dot{\psi} \Delta}{(1 + m \text{ cof. } \psi)^2}$ whose fluent we are investigating. That

we may the more readily separate this complex fluxion into its component terms, it will be convenient, for the present, to substitute x , and $1 - x^2$ for $\text{cof. } \psi$, and $\text{fin.}^2 \psi$ respectively, and then reduce the whole to a common denominator, it thus becomes

$$\frac{m^2 x^4 + 2e^2 x^3 + (1 - 2e^2)x + m(1 - e^2)}{(1 + mx)^2} \cdot \frac{\dot{\psi}}{\Delta},$$

and this again by methods of reduction sufficiently obvious is resolvable into

$$\left\{ \frac{-e^2}{m} (1 - x^2) + \frac{1}{m(1 + mx)} - \frac{1 - m^2}{m} \cdot \frac{1 - e^2(1 - x^2)}{(1 + mx)^2} \right\} \frac{\dot{\psi}}{\Delta};$$

now the last term of this expression is evidently the very fluxion proposed, which, it thus appears, depends upon the fluxion of an algebraic quantity, and other fluxions, whose fluents are to be farther the subject of investigation.

Let us now put $\frac{1 - m \text{ cof. } \dot{\psi}}{1 - m^2 \text{ cof.}^2 \dot{\psi}}$ for $\frac{1}{1 + m \text{ cof. } \dot{\psi}}$ and restore cof. $\dot{\psi}$, and fin.² $\dot{\psi}$ for x and $1 - x^2$, then putting $\dot{Z}$ for the whole expression, or for the fluxion of $\frac{\text{fin. } \dot{\psi} \Delta}{1 + \text{cof. } \dot{\psi}}$ we have by

transposition $(1 - m^2) \frac{\dot{\psi} \sqrt{(1 - e^2 \text{ fin.}^2 \dot{\psi})}}{(1 + m \text{ cof. } \dot{\psi})^2}$ equal to

$$- m \dot{Z} - \frac{m \dot{\psi} \text{ cof. } \dot{\psi}}{(1 - m^2 \text{ cof.}^2 \dot{\psi}) \Delta} + \frac{\dot{\psi}}{(1 - m^2 \text{ cof.}^2 \dot{\psi}) \Delta} - \frac{e^2 \dot{\psi} \text{ fin.}^2 \dot{\psi}}{\Delta}$$

The fluent of $\dot{Z}$ we already know; the second term

$\frac{m \dot{\psi} \text{ cof. } \dot{\psi}}{(1 - m^2 \text{ cof.}^2 \dot{\psi}) \Delta}$ may be integrated by an arch of a circle;

the two remaining terms, by putting $\frac{m^2}{1 - m^2} = n$ and substituting $1 - \text{fin.}^2 \dot{\psi}$ for $\text{cof.}^2 \dot{\psi}$, may be reduced to this form

$$\dot{\psi} \Delta - \frac{1 - n}{2} \cdot \frac{\dot{\psi}}{\Delta} + \frac{1 + n}{2} \cdot \frac{1 - n \text{ fin.}^2 \dot{\psi}}{1 + n \text{ fin.}^2 \dot{\psi}} \cdot \frac{\dot{\psi}}{\Delta}$$

where $\dot{\psi} \Delta$ and $\frac{1 - n}{2} \cdot \frac{\dot{\psi}}{\Delta}$ are integrable by the arcs of conic

sections: as to the fluxion $\frac{1 - n \text{ fin.}^2 \dot{\psi}}{1 + n \text{ fin.}^2 \dot{\psi}} \cdot \frac{\dot{\psi}}{\Delta}$ its fluent is in

general a transcendant of a higher order than can be expressed by arches of conic sections; there are cases, however, in which it may be expressed by an arch of a circle, and this happens in particular when $n = e$: Now, by the question, $d = \frac{1}{2} \sqrt{(3)}$, $p = \frac{1}{2}$, and, therefore, $m = \frac{1}{3} \sqrt{(3)}$, $e = \frac{1}{2}$, and $n = \frac{1}{2} = e$,
hence

hence it appears, that the simplification just now mentioned, takes place in the case under consideration.

Upon the whole it appears that, by substituting numbers for p , d , m , and e ,

$$\dot{z} = \frac{1}{2} \left\{ \frac{-1}{\sqrt{3}} \dot{Z} - \sqrt{3} \frac{\dot{\psi} \text{ cof. } \psi}{(3 - \text{cof.}^2 \psi) \Delta} + \dot{\psi} \Delta - \frac{1}{2} \cdot \frac{\dot{\psi}}{\Delta} + \frac{1}{2} \cdot \frac{1 - \frac{1}{2} \text{fin.}^2 \psi}{1 + \frac{1}{2} \text{fin.}^2 \psi} \cdot \frac{\dot{\psi}}{\Delta} \right\}$$

where $\Delta = \sqrt{(1 - \frac{1}{2} \text{fin.}^2 \psi)}$.

Now the fluent of $\dot{Z}$ is $\frac{\text{fin. } \psi \Delta}{1 + m \text{ cof. } \psi}$.

Take θ an arch of a circle such that $\text{tang. } \theta = \frac{1}{2} \sqrt{3} \frac{\text{fin. } \psi}{\Delta}$,

then, the fluent of $\sqrt{3} \frac{\dot{\psi} \text{ cof. } \psi}{(3 - \text{cof.}^2 \psi) \Delta}$ is the arch θ .

The fluent of $\dot{\psi} \Delta$ or $\dot{\psi} \sqrt{(1 - \frac{1}{2} \text{fin.}^2 \psi)}$ is evidently an arch of an ellipse, having its semi-conjugate axis $= 1$, and excentricity $= e = \frac{1}{2}$, the arch to be reckoned from the extremity of the conjugate axis to that of an ordinate to the transverse, whose distance from the centre is $\text{fin. } \psi$; let this arch be denoted by E .

Let ε denote the excentricity of another ellipse, such, that the semi-conjugate axis being also 1, $\varepsilon = \frac{1 - \sqrt{(1 - e^2)}}{1 + \sqrt{(1 - e^2)}} =$

$\frac{1 - \frac{1}{2} \sqrt{3}}{1 + \frac{1}{2} \sqrt{3}}$, and ω an arch of a circle, such that $\text{fin. } \omega =$

$\frac{\left\{ 1 + \sqrt{(1 - e^2)} \right\} \text{fin. } \psi \text{ cof. } \psi}{\sqrt{(1 - e^2 \text{fin.}^2 \psi)}}$, or so that $\text{tan. } (\omega - \psi) =$

$\sqrt{(1 - e^2)} \cdot \text{tan. } \psi$, where it is to be observed that while ψ increases from 0 to π , ω increases from 0 to 2π , let E' denote an arch of this ellipse, intercepted between the conjugate axis, and an ordinate to the transverse, the distance of the ordinate from the centre being $= \text{fin. } \omega$, then, putting $c (= \frac{1}{2} \sqrt{3})$ for the

semi-conjugate of the former ellipse, the fluent of $\frac{\dot{\psi}}{\Delta}$ or

$\frac{\dot{\psi}}{\sqrt{(1 - e^2 \text{fin.}^2 \psi)}}$ is $\frac{1}{c} \left\{ -E + \frac{1+c}{2} E' + \frac{1-c}{2} \text{fin. } \omega \right\}^*$

* See *Traité du Calcul Differential et du Calcul Integral par La Croix*, Vol. II, page 181; the same may be found from *Landen's Memoirs*.

Finally,

Finally, let τ be such an arch that $\frac{(1 + e) \sin. \psi}{1 + e \sin.^2 \psi}$ or

$\frac{3 \sin. \psi}{2 + \sin.^2 \psi} = \sin. \tau$, then the fluent of

$\frac{1 - \frac{1}{2} \sin.^2 \psi}{1 + \frac{1}{2} \sin.^2 \psi} \cdot \frac{\dot{\psi}}{\sqrt{(1 - \frac{1}{4} \sin.^2 \psi)}}$ is equal to $\frac{2}{3} \tau$

and these fluents being substituted in the general expression will give the area required.

The whole area, or the fluent generated, while ψ from 0 becomes 2π , will be found by considering that ψ and τ also increase from 0 to 2π , θ increases from 0 to a certain limit, and then decreases to 0, and ω increases from 0 to 4π ; hence $Z = 0$, $\theta = 0$, $\tau = 2\pi$, the ellipse whose excentricity is e increases from 0 to the whole circumference, and the ellipse whose excentricity is ε increases from 0 to two circumferences; hence, putting A for a quadrant of the former ellipse, and B for a quadrant of the latter, the whole area is

$$\frac{1}{2}\pi + (2 + \frac{1}{2c})A - (\frac{1}{2} + \frac{1}{2c})B = \frac{1}{2}\pi + (2 + \frac{1}{\sqrt{3}})A - (\frac{1}{2} + \frac{1}{\sqrt{3}})B$$

SECOND SOLUTION, by Mr. J. I.

Let r denote the radius of the base of the cone, and h the perpendicular drawn from the vertex to the plane of the base: Let z denote a variable arch of the circular base reckoned from a fixt origin, and let ρ denote the line drawn in the conical surface from the extremity of z , to the vertex: From the extremity of the indefinitely small arch $\dot{z}$, draw a perpendicular to ρ ; then that perpendicular will be one-leg of a right-angled triangle, of which $\dot{z}$ is the hypothenuse, and $\dot{\rho}$ the other leg: it will, therefore, be equal to $\sqrt{(\dot{z}^2 - \dot{\rho}^2)}$. Therefore, the fluxion of the conical surface bounded by ρ , or the area of the small triangle standing on the base $\dot{z}$, and bounded by lines drawn to the vertex of the cone, will be equal to $\frac{\rho}{2} \sqrt{(\dot{z}^2 - \dot{\rho}^2)}$. If now we put s for the conical surface sought, we have

$$\dot{s} = \frac{\rho}{2} \sqrt{(\dot{z}^2 - \dot{\rho}^2)}.$$

Draw a straight line from the foot of the perpendicular through the centre of the base, and let a , denote the part of it intercepted between the perpendicular and the centre of the base: Let ϕ be the

the measure of z on the circle having unity for radius, and suppose the origin of z to be first in the extremity of the diameter drawn through the foot of the perpendicular: Then, we readily get $\rho^2 = h^2 + (a - r \cos. \phi)^2 + r^2 \sin.^2 \phi = h^2 + a^2 + r^2$

$- 2ar \cos. \phi$: hence $\dot{\rho} = r\dot{\phi} \times \frac{a \sin. \phi}{\rho}$: But $\dot{z} = r\dot{\phi}$:

therefore, by substitution $\dot{s} = \frac{r\dot{\phi}}{2} \sqrt{(\rho^2 - a^2 \sin.^2 \phi)} = \frac{r\dot{\phi}}{2} \times$

$\sqrt{h^2 + (r - a \cos. \phi)^2}$. And putting $m = \frac{r}{h}$, $n = \frac{a}{h}$ we get

$$\dot{s} = \frac{r\dot{\phi}}{2} \times \dot{\phi} \sqrt{(1 + (m - n \cos. \phi)^2)}$$

Put now $y = m - n \cos. \phi$; then $n \sin. \phi = \sqrt{(n^2 - (m - y)^2)}$

And $\dot{\phi} = \frac{\dot{y}}{\sqrt{(n^2 - (m - y)^2)}}$: therefore,

$$\frac{\dot{s}}{rh} = \frac{\dot{y} \sqrt{(1 + y^2)}}{\sqrt{(n^2 - (m - y)^2)}}$$

It would far exceed our limits to treat the fluxional expression just obtained in a general manner: we shall, therefore, confine our attention to the particular case in the question. We have

then $m = 2$, $n = \sqrt{3}$, and $rh = \frac{r^2}{2}$, therefore, having substituted, the fluxion to be integrated becomes

$$\frac{4\dot{s}}{r^2} = \frac{\dot{y} \sqrt{(1 + y^2)}}{\sqrt{(4y - y^2 - 1)}}$$

where $y = 2 - \sqrt{3} \cos. \phi$, and, therefore, it is contained in the limits $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

Let ϕ' denote the measure of another arch, reckoned from the opposite extremity of the diameter that ϕ is, and so related to ϕ that $(2 + \sqrt{3} \cos. \phi) \times (2 - \sqrt{3} \cos. \phi) = 1$: then $2 +$

$\sqrt{3} \cos. \phi' = \frac{1}{y}$: also, let s' denote the conical surface, insisting on ϕ' , then we shall obviously have

$$\frac{2\dot{s}'}{r^2} = \frac{\dot{y} \sqrt{(1 + y^2)}}{y^2 \sqrt{(4y - y^2 - 1)}}$$

Put $y + \frac{1}{y} = 2(u + 1)$: then $y = u + 1 + \sqrt{(u^2 + 2u)}$:

also $\frac{1}{y} = u + 1 - \sqrt{(u^2 + 2u)}$: and $\frac{y^2 + 1}{4y - y^2 + 1} = \frac{u+1}{1-u}$:
therefore,

therefore,

$$\frac{\dot{s}}{r^2} = \frac{\dot{u} + u\dot{u}}{\sqrt{[1-u^2]}} + \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}}$$

$$\frac{\dot{s}'}{r^2} = -\frac{\dot{u} + u\dot{u}}{\sqrt{[1-u^2]}} + \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}}$$

whence $\frac{2}{r^2}(\dot{s} + \dot{s}') = \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}}$. And

$\frac{2}{r^2}(\dot{s} - \dot{s}') = \frac{\dot{u} + u\dot{u}}{\sqrt{[1-u^2]}}$. In these formulæ

$u = \frac{(y-1)^2}{2y} = \frac{(1-\sqrt{[3]}\operatorname{cof}.\varphi)^2}{4-2\sqrt{[3]}\operatorname{cof}.\varphi}$; and, therefore, u is con-

tained in the limits 1 and 0. The latter formula is evidently integrable, and we get $\frac{2}{r^2} \times (s-s') = -\frac{\pi}{2} + \psi - \operatorname{cof}.\psi$;

where $\sin.\psi = u$, and $\pi = 3.1415$ &c.; from which we learn, that the difference of the two conical surfaces s and s' is equal to a sector of the base, together with a rectilinear space: and it will be a maximum when $u = 0$, or when $\operatorname{cof}.\varphi =$

$$\frac{1}{\sqrt{[3]}} \text{ and } \operatorname{cof}.\varphi' = -\frac{1}{\sqrt{[3]}}.$$

Let us now consider the second formula

$$\frac{2}{r^2}(\dot{s} + \dot{s}') = \frac{\dot{u}(u+1)^2}{\sqrt{[(1-u^2)(u^2+2u)]}} = \frac{\dot{u}(u+1)^2}{\sqrt{[(u^2+u)(2-u-u^2)]}}$$

Assume $u^2 + u = 2z^2$: then $u = \frac{1}{2}\left\{\sqrt{[8z^2+1]}-1\right\}$:

And $\dot{u} = \frac{4z\dot{z}}{\sqrt{[8z^2+1]}}$: whence by substitution

$$\frac{2}{r^2}(\dot{s} + \dot{s}') = \frac{-\dot{z}(4z^2+1)}{\sqrt{[(1-z^2)(8z^2+1)]}} - \frac{\dot{z}}{\sqrt{[1-z^2]}}$$

The fluent of the latter part is obviously nothing more than the arch of which z is the sine: Let $\operatorname{cof}.\psi = z = \sqrt{\frac{u^2+u}{2}} =$

$\frac{1-y}{2y} \cdot \sqrt{\frac{1+y^2}{2}}$, and while y increases from $2 - \sqrt{3}$ to

$2 + \sqrt{3}$, z will decrease from $+1$, to -1 : then, integrating,

$$\frac{2}{r^2}(s + s') = \int \frac{-\dot{z}(4z^2+1)}{\sqrt{[(1-z^2)(8z^2+1)]}} dz + \psi'$$

The whole is now reduced to the integration of one formula, which I shall call $\dot{H}$, and which is comprehended in the forms

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that Mr. Landen has integrated by means of elliptic peripheries.

Assume $\sqrt{[8z^2 + 1]} = x + \sqrt{[3 + x^2]}$ then $\frac{4z \sqrt{[1 - z^2]}}{\sqrt{[8z^2 + 1]}}$

$$= \sqrt{[1 - x^2]} \text{ and } \frac{8z\dot{z}}{(8z^2 + 1)} = \frac{\dot{x}}{\sqrt{[3 + x^2]}}$$

$$\text{therefore, } \frac{\dot{x}}{\sqrt{[(1 - z^2)(8z^2 + 1)]}} = \frac{\frac{1}{2}\dot{x}}{\sqrt{[(1 - x^2)(3 + x^2)]}}$$

$$\text{But, } H = \frac{2}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \sqrt{[8z^2 + 1]} + \frac{1}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \\ \times \left\{ \sqrt{[8z^2 + 1]} + \frac{3}{\sqrt{[8z^2 + 1]}} \right\} = \frac{2}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}}$$

$$\sqrt{[8z^2 + 1]} + \frac{1}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}} \times 2\sqrt{[3 + x^2]} = \frac{2}{6} \cdot \frac{-\dot{z}}{\sqrt{[1 - z^2]}}$$

$$\sqrt{[8z^2 + 1]} + \frac{1}{6} \cdot \frac{-\dot{x}}{\sqrt{[1 - x^2]}} \sqrt{[3 + x^2]} + \frac{1}{6} \cdot \frac{-x\dot{x}}{\sqrt{[1 - x^2]}}$$

where the fluents of the two first parts are obviously elliptic arches, and the third part is an algebraic quantity. Let $x = \text{cof. } \psi''$

$$H = \int \dot{\psi} \sqrt{[1 - \frac{8}{9} \sin.^2 \psi]} + \frac{1}{2} \int \dot{\psi}'' \sqrt{[1 - \frac{1}{4} \sin.^2 \psi]} + \frac{\sin. \psi''}{6} \text{ where the two first parts are arches of two ellipses of}$$

which the excentricities are $\frac{2\sqrt{2}}{3}$ and $\frac{1}{2}$, the semi-transverse of each being 1: the arches ψ' and ψ'' are reckoned from the extremity of the conjugate axis, and are defined by ordinates parallel to the transverse equal to $\sin. \psi'$ and $\sin. \psi''$ respectively.

The preceding analysis enables us to assign a portion of the conic surface standing on an indefinite arch of the base, as we have computed both the sum and difference of the parts denoted by s and s' . But we are required only to find the whole surface of the cone which is now readily done, and is equal to

$$\frac{1}{4} \left\{ E + \frac{2}{3} E' \right\} + \frac{\pi}{2}$$

where E denotes the periphery of the ellipse of which the excentricity is $\frac{2\sqrt{2}}{3}$; E' that of the ellipse of which the excentricity is $\frac{1}{2}$; and $\pi = 3.1416$.

Note. It may be proper to remark that the arch ψ'' becomes a semicircle when ψ' becomes a quadrant.

XVII. QUESTION 17, by GEOMETRICUS.

A triangle being given by position, and two points being also given, it is required to draw lines through these points to intersect each other in one of the sides of the triangle and meet the other sides so as to cut off segments adjacent to given points in them, having to each other a given ratio.

FIRST SOLUTION, by Mr. W. WALLACE, *R. M. College.*

ANALYSIS. Fig. 25, Pl. 1. Let ABC be the triangle given by position, P, Q the given points, from which lines are to be drawn to meet in BC , one of its sides, and F, G the given points in AB, AC the remaining sides. Suppose the problem resolved, that is, suppose a point D is found in BC , such, that DP, DQ being drawn and produced to meet the given lines AB, AC in H and K respectively, the ratio of FH to GK may be given. Take GL to FB in the given ratio of GK to FH , thus L will be a given point, and the ratio of HB to KL the same with the given ratio of FH to GK . Draw PM parallel to AB meeting BDC in M ; the line PM will evidently be given by position, therefore, M will be a given point, and a line drawn from Q to M will be given by position. Draw BR parallel to MQ , meeting DQ in N , then DR being drawn through a given point parallel to a line given by position, will also be given by position.

The triangles DMQ, DBN are similar, hence $BN : MQ :: BD : DM$, but the triangles DMP, DBH , being also similar $BD : DM :: BH : MP$, therefore, $BN : MQ :: BH : MP$, and alternately, $BN : BH :: MQ : MP$; now, MQ, MP are given lines, for the points M, P, Q are given, therefore, the ratio of BN to BH is given, but by hypothesis and from what has been already shewn, the ratio of BH to LK is given. Thus it appears, that B and L are given points in BR, LA lines given by position, and that QKN is a line drawn from Q a given point without them, cutting off segments BN, LK adjacent to the given points B, L , and having to each other a given ratio; now this is a problem well known, and which has been constructed in various ways by writers on Geometry, for example by Simpson, in his *Elements of Geometry* (Prob. 37, page 242, second edit.) but more especially by Dr. Halley, in his *Translation of the Sectio Rationis* of Apollonius.

The manner of construction and demonstration follow so obviously from the preceding analysis, that, for the sake of brevity, we omit them.

SECOND SOLUTION, *by Mr. JOHN BUTTERWORTH.*

Let LMN (fig. 26, pl. 1.) be the given triangle, D and C the given points in the sides MN and LN, and A and B the other given points. Suppose EH and EG to be drawn through the points A and B to meet the sides LN and MN in H and G so that CH may have to DG the given ratio. From A draw AS parallel to LN meeting LM in S, join BS, and draw BA to meet LN at I and CA to meet LM at P, and join PB; on BS take BT = IC, and draw TK parallel to AB to meet BP in K; also KR parallel to SB to meet MN at R. Now, it is manifest, from Prop. IV. of Professor Playfair's Paper on Prisms *, that if any two lines whatever, as AE, BE, be inflected from the points A and B to the line LM and meet KR in F and LC in H, KF will be equal to HC. Hence the problem is reduced to this, viz. Through the given point B to draw a right line GF to meet MN and KR at G and F, so that DG and KF may have a given ratio; the method of doing this is pointed out at page 25, vol. III. of the Old Series of the Repository.

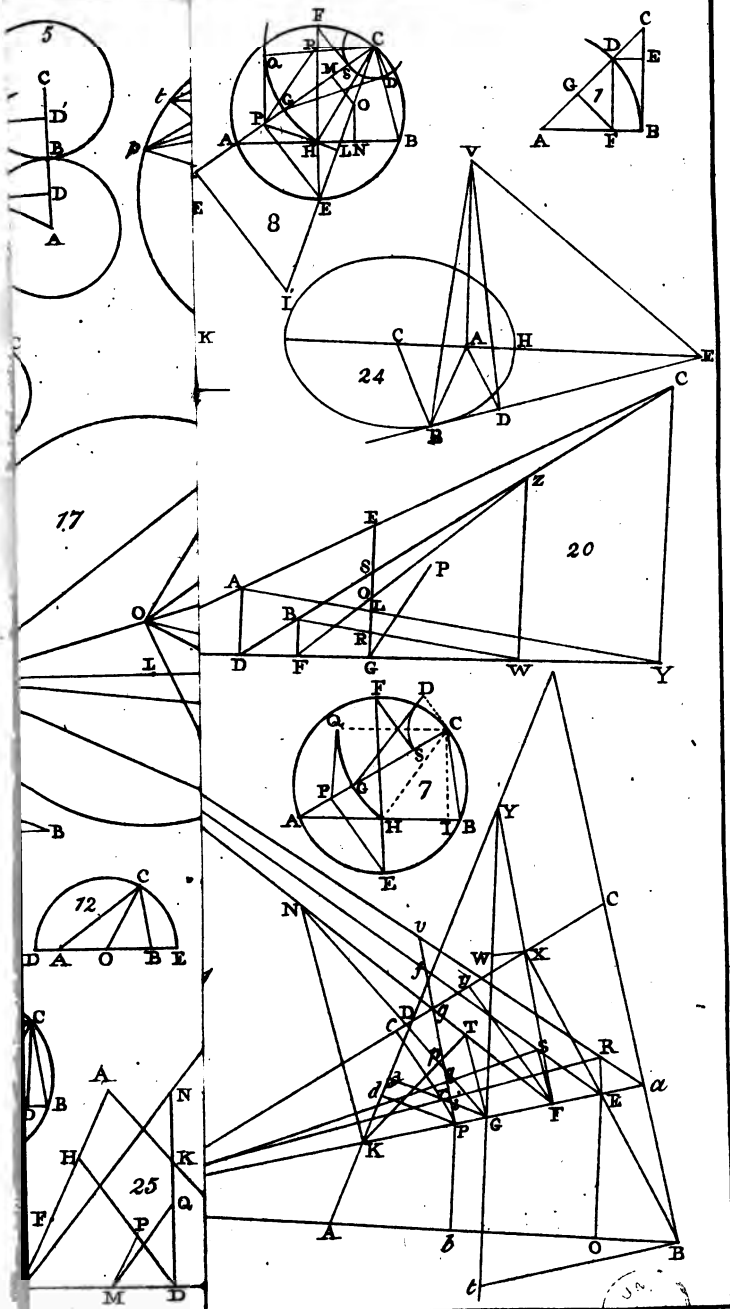
XVIII. QUESTION 18, *by GEOMETRICUS.*

Let A, B, be two given points in AB a straight line given by position, and let C, D be two given points without that line; let CV, DV, be drawn meeting AB in F and G, so that AF may have to BG a given ratio; It is required to determine the Locus of the point V.

FIRST SOLUTION, *by Mr. W. WALLACE, R. M. College.*

ANALYSIS. Divide AB at H (fig. 27, 28, 29, 30, pl. 2.) so that AH may have to HB the given ratio of AF to BG, then H

* This Paper is printed in the First Part of the Mathematical Tracts and Selections.





will be a given point, and FH will have to HG the given ratio that AF has to BG, or that AH has to HG. Now there will be several cases of the problem, depending upon the position of the given points C, D with respect to the line AB and to the point H.

CASE I. Suppose that the line which joins the points C, D is parallel to AB (fig. 27.); join VH meeting CD in K, then because $FH : HG :: CK : KD$, the ratio of CK to KD is given; therefore, the point K, and the line KH are given by position, so that in this case the locus of V is a straight line given by position.

CASE II. Suppose that the line which joins the points C and D passes through H (fig. 28.); draw FK parallel to DG meeting HD in K, then $HF : FG :: HK : KD$, thus the ratio of HK to KD is given, and, therefore, K is a given point. Again, $CK : KD :: CF : FV$, but CK and KD are given magnitudes, therefore, the ratio of CF to FV is given, now C is a given point, and F is in a straight line AB given by position, therefore, the locus of V in this case is also a straight line given by position, which will be parallel to the given line AB.

CASE III. Next suppose that the line which joins the points C, D meets AB in E, (fig. 29.) so that the given ratio of FH to HG is the same as that of CE to DE, and so that HF, HG lie in the same, or in a contrary direction, according as CE, DE lie in the same, or in a contrary direction. Draw CM parallel to HG meeting DV in N. The ratio of CN to HG is compounded of the ratios of CN to FG and of FG to HG; but $CN : FG :: VN : VG$ and by hypothesis $FG : HG :: CD : DE :: ND : DG$, therefore, the ratio of CN to HG is compounded of the ratios of VN to VG, and of ND to DG, hence $CN : HG :: VN \times ND : VG \times DG$, thus it appears, that the locus of V is a parabola which passes through three given points H, C, D, and has its axis parallel to HG a line given by position, therefore, the parabola itself is given by position.

CASE IV. In the last place, let us suppose that the point H and the line which joins the given points C, D have any other positions than those already considered. Join HK (fig. 30.) and draw FK parallel to DG meeting DH in K: then $DH : HK :: GH : HF$, that is in a given ratio, now D and H are given points, therefore, K is also a given point. Join CK, this line will be given by position, and will meet AB in a given point L, for otherwise CK must be parallel to AB, and, therefore, producing DC to meet AB in E, $CE : ED :: KH : HD :: FH : HG$, which is contrary to the hypothesis. Draw HM, VQ parallel to CL, meeting FV, FE in N and Q, the line HM will evidently be given by position.

The triangles QFV, LFC are similar, also the triangles QGV, LFK, hence

QF

$QF : QV :: FL : LC$
 and $QV : QG :: KL : LF$
 therefore, ex aeq. $QF : QG :: KL : LC$

Now KL, LC are given magnitudes, therefore, the ratio of $QF : QG$, and hence the ratio of QF to FG is given, but by hypothesis the ratio of GF to FH is given, therefore, the ratio of QF to FH , or that which is equal to it, the ratio of VF to FN is given, so that the ratio of FV to NV is also given. We have now found that from the given point C there is drawn a line CV to meet two lines HM, HA given by position, so that FV, NV the distances of V from the lines given by position have to each other a given ratio, hence the locus of V is a given hyperbola as is pretty generally known, and as was perhaps first shewn by Dr. Barrow, in his Geometrical Lectures (Lect. VI.)

SECOND SOLUTION, by Mr. J. I.

Divide AB in E (fig. 31, pl. 2.) so that $AE : EB :: AF : AG$. Join CE and make $CE : EH :: AF : FG$ or $AE : EB$. Draw DLH and CK parallel to it. Then, because $AE : EB :: AF : GB$; therefore, $AE : EB :: FE : EG :: KE : EL$; consequently, $KF : GL :: KE : EL$, or as CK to HL . Draw CR parallel to AB and divide DR in N , so that $DN : NR :: DL : LH$; draw $NQSP$ parallel to AB meeting DV in Q , CV in P , and KC in S . Also divide KL in M , so that $KM : ML :: DL : LH$. And draw VO, MT parallel to DH . Then, from what has been shewn,

$$HL : LG :: CK : KF :: CS : SP,$$

$$\text{but } LD : LH :: DN : NR \text{ or } CS;$$

$$\text{therefore, } LD : LG :: DN : PS :: DN : NQ;$$

$$\text{therefore, } PS = NQ.$$

$$\text{Again, } PS : CS \text{ or } NR :: PO : OV,$$

$$\text{and, } DN : NQ \text{ or } PS :: VO : OQ;$$

$$\text{therefore, } DN : NR :: PO : OQ :: DL : LH :: PM : ML.$$

But $PS = NQ$; therefore, $PQ = NS = KL$; therefore, $PO = KM, OQ = ML$, and, $OT = PS = NQ$.

Because now $OT = PS$, and $OP = TS$; therefore, $VO : PO$ or $ST :: CS : SP$ or OT , and $VO \times OT = CS \times ST$.

Consequently the locus required is an hyperbola having TM and TN for its asymptotes and passing through the point C .

The construction is manifest from the Analysis. It may be remarked

remarked that the locus is a straight line when C and D are in a straight line parallel to AB. The locus is also a straight line when the three points C, E, D are in a straight line. And when the line DH is parallel to AB the locus is no longer a hyperbola but a parabola.

THIRD SOLUTION, *by Mr. WILLIAM SMITH, Liverpool.*

CONSTRUCTION. Divide the given line AB (fig. 32, pl. 2.) in E, so that AE may be to EB in the ratio, which, by the question AF is required to have to BG, and divide CD in e in the same ratio; join Ee and draw CA', DB' parallel thereto, meeting AB in A' and B'. Join DE, which produce to L, so that DE : DL :: DB' : A'C; through L draw an indefinite line parallel to Ee; draw CZ parallel to DL, meeting the parallel to Ee, through L, in Z; from D, on the opposite side of D to L, set off DM = LE, and join CM meeting LZ in H. Produce CZ to P', so that the rectangle ZP'·ZL may be equal to the rectangle DL·LE; take the point P so that ZP : ZP' :: HL : LM and complete the parallelogram ZHUP; find the point g in CA' such that the rectangle Cg·CH may be to ZP·ZH :: CH : CZ, and from g draw gO parallel to CH meeting PU in O; then with the centre O and asymptotes gO, OP describe an hyperbola whose power may be equal to the rectangle gO·gC, and it will be the locus required.

DEMONSTRATION. From D and C to any point V in the hyperbola draw CV (meeting AB in F, DL in f, HL in n, PU in i, and Ee in R) and DV (meeting AB in G, Ee in r, and HL in S), join SR and LV meeting SR in I and from S draw SK parallel to CV meeting DL in K; draw VQdW parallel to DL (meeting PO in Q, HL in d, and CA in W) and Vbo parallel to CH meeting PO in b and CA' in o, and through n draw plqN (meeting CA' in p, PO in t, a line through V parallel to PO in q and CM in N). Then, by construction, bg·gO = Ob·bV, that is, by parallels, &c. og : ob :: Cg : bV, hence, bg is parallel to CV, and, therefore, bi (=Cg) a given line. By the similar triangles CHn, Vib, Hn·bV (=CH·ib) is given = CH·bg; therefore, since bV : tq :: CH : CZ, and also, by construction, (CH·Cg) Hn·bV : HZ·ZP :: CH : CH, it follows that Hn·tq = HZ·ZP, that is, the figure Utqk = ZHUP, add to both HntV, and Zn·nt = Hn·dV(nq); but by construction, nt (ZP) : ZP' :: Hn : nN, therefore, Zn·ZP = Nn·dV, but by parallels Zn : nN :: ZL : Mf, therefore, ZL·ZP' = Mf·dV, also by construction, ZL·ZP' = DL·LE, therefore, DL·LE = Mf·dV, but by parallels, DL : LK :: VD : fK (Vc), therefore,

fore, $DK \cdot dV = DL \cdot fK$. Whence, since $DD \cdot LE = Mf \cdot dV$ and $DL \cdot fK = DK \cdot DV$, $LE (MD) : fK :: Mf : DK$, or, dividendo et invertendo, $MD : Df :: fK : Df$, therefore, $MD (= EL) = fK$; but since from the points S, R , two parallels intersect Dm in f, K , and make the intercapted parts EL, fK equal to each other; therefore, the line joining the points S, R is parallel to Dm , therefore, $ER = SL$, whence, $Er : ER (SL) :: DE : DL ::$ (by constr.) $DB' : CA'$, that is, $DB' : Er :: A'C : ER$, or $B'G : GE :: A'F : FE$, or $B'G : A'F :: B'E : EA' :: BE : EA$; that is, BG is to AF in the ratio required.

Q. E. D.

Mr. Marrat sent an algebraical solution to this question.

XIX. QUESTION 19, by Mr. WILLIAM WALLACE, of the Royal Military College, Great Marlow, Bucks.

Let AB be the diameter of a circle, and FG a straight line perpendicular to the diameter, meeting it in C , draw any two lines CHD, CKE meeting the circle in H, D , also in K, E ; join DK, EH , meeting the line perpendicular to the diameter in F and G , then FC is equal to CG : Required the demonstration?

FIRST SOLUTION, by Mr. THOMAS BAZLEY, Bolton.

CASE I. When the point C is within the circle. Fig. 33, Pl. 2.

Let the points where FG meet the circle be M, N ; through G draw PQG parallel to KD meeting KE in Q and DH in P . Then by the similar triangles $QEP, HPG, QG : EG :: HG : PG$; therefore, $GE \times GH = GP \times GQ$; and by the similar triangles QGC, KFC , we have, $KF : QG :: CF : CG$, and, $FD : GP :: CF : CG$. Hence, $KF \times FD : QG \times GB = GE \times GH :: CF^2 : CG^2$.

But by the property of the circle,
 $GE \times GH = MG \times GN = CN^2 - CG^2$,
 and $KF \times FD = MF \times FN = CM^2 - CF^2$;

consequently,

consequently, $CM^2 - CF^2 : CN^2 - CG^2 :: CF^2 : CG^2$,
and by composition, $CM^2 : CN^2 :: CF^2 : CG^2$.

But $CM = CN$, because BC is perpendicular to MN ; therefore, $CF = CG$, which was to be proved.

CASE II. When the point C is without the circle. Fig. 34, Pl. 2.

In the very same manner as in the first Case we shall have, $KF \times FD : GE \times GH :: CF^2 : CG^2$; now, through the centre O draw OG and OF , meeting the circle in I and L . Then by the property of the circle,

$KF \times FD = OF^2 - OL^2$, and $GE \times GH = OG^2 - OI^2$; therefore,
 $OF^2 - OL^2 : OG^2 - OI^2 :: CF^2 : CG^2$, and by division,
 $OF^2 - OL^2 - CF^2 : OG^2 - OI^2 - CG^2 :: CF^2 : CG^2$;
but $OF^2 - CF^2 = OC^2 = OG^2 - CG^2$; therefore,
 $OC^2 - OL^2 : OC^2 - OI^2 :: CF^2 : CG^2$;

but the two first terms of this proportion are evidently equal; and, therefore, $CF^2 = CG^2$, or $CF = CG$.

W. W. D.

SECOND SOLUTION, by Mr. WALLACE, the Proposer.

This question, owing to an accidental circumstance, was proposed in another place. I shall here give it a more general form, and shew that the same property belongs to any conic section whatever.

THEOREM. Let C (Fig. 35, Pl. 2.) be any point in the diameter of a conic section; MN a straight line passing through C parallel to a tangent at the extremity of the diameter, CHD , CKE , any two straight lines passing through C and meeting the curve in the points D , H , and E , K , join EH , DK , meeting MN in G and F , then CG is equal to CF .

Suppose the point C to be without the conic section, draw CP , CQ tangents to the curve at P and Q , and join PQ which will be parallel to MN , draw tangents to the curve at the points D , H , and E , K , let R and S be the intersections of the tangents, and L the intersection of the lines DK , EH , the points R , L , S , are in a straight line; (See Robertson's Treatise on Conic Sections, Book IV. Prop. 18.) but the points R , S , are in the line PQ produced (Robertson, Book IV. Prop. 17.), therefore, the point L is in the line PQ .

Let CD meet PQ in T , then CD is harmonically divided at

the points C, H, T, D, (Robertson, Book IV. Prop. 16.) or $DT : DC :: TH : HC$, but because TL is parallel to GF, $DT : DC :: TL : CF$, also $HT : HC :: TL : CG$, therefore, $TL : CF :: TL : CG$, and $CF = CG$.

When C is within the curve, then L will fall without it, and it will readily appear, from the prop. already quoted, that if tangents be drawn at the points in which MN cuts the curve, the point L will be in a straight line passing through their intersection parallel to MN. If DC be produced to meet the parallel in T, the remainder of the demonstration will be as before.

Solutions were likewise given by Messrs. Butterworth, Collins, Johnson, Morley, Pickering, and Wright.

XX. QUESTION 20, by Mr. WILLIAM SMITH, *Liverpool.*

Given the foci of an ellipsis to describe it so as to touch a circle given in position and magnitude.

SOLUTION, by Mr. THOMAS HOPPER.

Suppose the problem solved, and from the given foci conceive right lines drawn to the point where the circle and ellipsis are in contact; then, it is manifest that a tangent to the ellipsis at this point will be perpendicular to a right line drawn from it to the centre of the circle; which right line or radius will, therefore, bisect the angle formed by the two focal lines.

Hence the Problem is reduced to that of drawing, from two given points, two right lines to meet each other in the periphery of a given circle, so that if another right line be drawn from the point of concurrence to the centre of the circle, it shall make equal angles with the lines first drawn. Which has been done, *vide* page 223, of the Appendix to the 2d. Edit. of Simson's Conic Sections; *vide* also, Examp. 7, Book 10, De l'Hospital's Conics.

And thus the Question was answered, by Mr. Wright.

XXI. QUESTION 21, by Mr. JOHN LOWRY.

Having given any number of points A, B, C, D, &c. in the periphery of a given circle it is required to find another point P (in the periphery) such that joining PA, PB, PC, &c. the sum of the lines PA, PB, PC, &c. may be equal to a given line.

FIRST SOLUTION, by Mr. WM. SMITH, *Liverpool*.

ANALYSIS. Fig. 36, Pl. 2. Suppose the thing done and that P is the required point; join PA, PB, PC, PD, &c. Produce AP so that PX may be equal to PB, and join BX, then, the angle AXB being equal to half the angle APB is given; therefore, the locus of the point X is a circle given in position described on AB; let XQ drawn parallel to CP meet the circle ABX in Q; then, since the angle AXQ ($=APC$) is given, Q is a given point; from C draw CL parallel to AX meeting QX in L, then, the angle CLQ ($=APC$) is given; therefore, the locus of the point L is a circle described on CQ and given in position. Produce PX to Y so that $XY = XL = CP$, and join YL meeting the circle CQL in R, then the angle QLR ($=\frac{1}{2}AXL = \frac{1}{2}APC$) is given, and R a given point; also, the angle AYR ($=\frac{1}{2}AXL = \frac{1}{2}APC$) is given, and the locus of the point Y is a circle given in position, passing through the points A, R, Y, and let YS drawn parallel to PD meet the circle AYR in S, then the angle AYS ($=APD$) is given, and S a given point; draw DM parallel to AY meeting SY in M, then the angle DMS ($=APD$) is given, and the locus of M a circle given in position and described on DS; take $YZ (=YM) = DP$, and join ZM meeting the circle DMS in T, then the angle SMT ($=\frac{1}{2}APD$) is given, and T a given point; also, the angle AZT ($=\frac{1}{2}AYM = \frac{1}{2}APD$) is given, and the locus of the point Z a circle given in position described on AT; but (supposing the number of points four, as it is evident the reasoning employed in the above Analysis, will apply, whatever the number of points may be) since AZ ($=AP + BP + CP + DP$) is a given line, the locus of Z is also a circle described from the centre A, and intersection of this with the former locus ATZ will determine the point Z. Whence this

CONSTRUCTION. On AB describe a segment of a circle to contain an angle equal to half the angle APB, and apply the chord AQ in that segment to subtend an angle equal to the angle APC subtended by AC in the given circle; on CQ describe a segment of a circle to contain an angle equal to APC, in which segment find the point R such that the angle subtended by QR in the circle CQ may be equal to half the angle APC; on AR describe a segment of a circle to contain an angle equal to half the angle APC, equal to half the angle subtended by RQ in the circle CRQ, in this segment apply the chord AS, to subtend an angle equal to the angle APD, and join DS, upon which describe a segment of a circle to contain an angle equal to the angle APD subtended by AD, in the given circle, in this segment find the point T, so that the angle subtended by TS may be equal to half the angle APD, let AT be joined, upon which let a segment of a circle be described to contain an angle equal to half the angle APD to which, from A, apply AZ equal to the given sum, meeting the circle ABCD in P, join AP, BP, CP, DP, and the thing is done.

SECOND SOLUTION, by Mr. JOHN WRIGHT, Norley.

ANALYSIS. Fig. 37, Pl. 2. Suppose the thing done, and AP, BP, CP, &c. drawn. Bisect the arc AB in Q, and join PQ, AQ, and AB. Then it is well known that $AP + BP : PQ$ in the given ratio of AB to AQ; therefore, when the number of given points is two, and $AP + BP$ given, PQ is given; and consequently the locus of the required point P in this case is a circle of which the centre is Q and radius QP.

Draw the chord CQ, and having determined the point R in the arc CBQ, so that the chords CR, QR may obtain the given ratio of AB to AQ, join CP; then, because $CR : QR :: AB : AQ :: PA + PB : PQ$, the rectangle $(PA + PB) \cdot QR = PQ \cdot CR = PR \cdot CQ - PC \cdot QR$ (Emerson's Geom. IV. 32.); whence $(PA + PB + PC) \times QR = PR \times CQ$, or $PA + PB + PC : PR :: CQ : QR$ a given ratio; hence, when the number of given points is three, and $PA + PB + PC$ given, PR is given, and the locus of P is a circle of which the centre is R and radius RP.

Draw the chord DR, and in the arch DRC let the point S be so taken, that chords DS, RS may obtain the given ratio of CQ to RQ, and join PS. Since $DS : RS :: CQ : RQ :: PA + PB + PC : PR$, the rectangle $(PA + PB + PC) \cdot RS = PR \cdot DS = PS \cdot DR - PD \cdot SR$; hence $(PA + PB + PC + PD) \cdot RS = PS \cdot DR$,

PS·DR, or PA + PB + PC + PD : PS :: DR : RS a given ratio; therefore, when the number of given points is four, and AP + BP + CP + DP given, PS is given, and the locus of P is a circle whose centre is S and radius SP. And thus the solution may be extended to any number of points.

THIRD SOLUTION, by Mr. W. WALLACE, R. M. College.

Let the given arches included between the points A, B, C, D, &c. be denoted by 2α , 2β , 2γ , &c. put 2ϕ for the arch PA, r for the radius of the circle, and a for the given straight line equal to the sum of the chords PA, PB, PC, PD, &c. which chords being equal to $2 \sin. \phi$, $2 \sin. (\alpha + \phi)$, $2 \sin. (\beta + \phi)$, $2 \sin. (\gamma + \phi)$, &c. respectively, we have,

$$2 \left\{ \sin. \phi + \sin. (\alpha + \phi) + \sin. (\beta + \phi) + \sin. (\gamma + \phi) + \&c. \right\} = a,$$

or, since $r \times \sin. (\alpha + \phi) = \sin. \alpha \cos. \phi + \cos. \alpha \sin. \phi$, &c.

$$\left. \begin{aligned} &\frac{2 \cos. \phi}{r} (\sin. \alpha + \sin. \beta + \sin. \gamma + \&c.) \\ &+ \frac{2 \sin. \phi}{r} (r + \cos. \alpha + \cos. \beta + \cos. \gamma + \&c.) \end{aligned} \right\} = a.$$

Put $2 (\sin. \alpha + \sin. \beta + \sin. \gamma + \&c.) = s$,
and $2 (r + \cos. \alpha + \cos. \beta + \cos. \gamma + \&c.) = c$;
then $s \cos. \phi + c \sin. \phi = ar$.

Take μ such an arch of the given circle that $\tan. \mu = \frac{rs}{c}$,

thus μ will be given, and $s = c \times \frac{\sin. \mu}{\cos. \mu}$, hence, by substitution and reduction,

$$\sin. \mu \cos. \phi + \cos. \mu \sin. \phi = \frac{ar}{c} \times \cos. \mu = \frac{ar}{c} \times \frac{r^2}{\sqrt{(r^2 + \tan.^2 \mu)}}$$

$$= \frac{ar^2}{\sqrt{(c^2 + s^2)}}.$$

$$\text{That is, } \sin. (\mu + \phi) = \frac{ar}{\sqrt{(c^2 + s^2)}}.$$

Hence

Hence the arch ϕ is determined, and since $\sin. (\mu + \phi)$ cannot exceed r , therefore, the given line a must not exceed $\sqrt{(c^2 + s^2)}$.

Ingenious Solutions were also received from Messrs. Butterworth, and Johnson.

XXII. QUESTION 22, by AMICUS.

It is well known, that if a denote an arch of a circle, radius being unity, the tangent of that arch is expressed by the series $a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c.$ but it is not so generally known what is, the scale of relation of the numeral coefficients of the terms : It is therefore required to determine that scale.

FIRST SOLUTION, by Mr. J. I.

Let $\tau = \tan. x$: then, $\frac{\tau}{x} = \frac{1}{\cos.^2 x} = \frac{2}{1 + \cos. 2x}$: or $\frac{\tau}{x}$

$\times \frac{1 + \cos. 2x}{2} = 1.$ Now

$$\frac{1 + \cos. 2x}{2} = 1 - \frac{2}{1.2} \cdot x^2 + \frac{2^3}{1.2.3.4} \cdot x^4 - \frac{2^5}{1.2.3.4.5.6} \cdot x^6 + \&c$$

And let τ be assumed $= x + \frac{A^{(3)}}{1.2} \cdot \frac{x^3}{3} + \frac{A^{(5)}}{1.2.3.4} \cdot \frac{x^5}{5} +$

$\frac{A^{(7)}}{1.2.3.4.5.6} \cdot \frac{x^7}{7} + \&c.$ in which the law of the denominators

is obvious, and the symbols, $A^{(3)}$, $A^{(5)}$, $A^{(7)}$, &c. denote indeterminate quantities of which the law of derivation is to be ascertained ; the numbers joined to the symbols mark the term of the series to which each belongs, by pointing out the exponent of x in that term ; then substituting the foregoing series in

in the equation investigated above; and putting the coefficients of the several powers of $x = 0$, we shall readily find,

$$A^{(3)} = 2,$$

$$A^{(5)} = \frac{4 \cdot 3}{1 \cdot 2} \cdot 2A^{(3)} - 2^3,$$

$$A^{(7)} = \frac{6 \cdot 5}{1 \cdot 2} \cdot 2A^{(5)} - \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^3 A^{(3)} + 2^5$$

$$A^{(9)} = \frac{8 \cdot 7}{1 \cdot 2} \cdot 2A^{(7)} - \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^3 A^{(5)} + \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 2^5 A^{(3)} - 2^7$$

or in general

$$A^{(2n+1)} = \frac{2n \cdot 2n-1}{1 \cdot 2} \cdot 2A^{(2n-1)} - \frac{2n \cdot 2n-1 \cdot 2n-2 \cdot 2n-3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 2^3 A^{(2n-3)} \\ + \frac{2n \cdot 2n-1 \cdot 2n-2 \cdot 2n-3 \cdot 2n-4 \cdot 2n-5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 2^5 A^{(2n-5)} - \&c.$$

In this formula, consisting of terms alternately positive and negative, each term is formed by multiplying together the corresponding terms of three other series, viz :

The series of quantities $A^{(2n-1)}$, $A^{(2n-3)}$, &c. reckoned backward which precede the quantity sought in the series of coefficients :

The series of the odd powers of 2; 2, 2^3 , 2^5 , &c. and the third series is nothing more than the 3d, 5th, 7th, &c. coefficients in the binomial $(a+b)^{2n}$

The law of continuation of the series for the tangent is therefore completely ascertained : And the method here given has the advantage of bringing out the series $A^{(3)}$, $A^{(5)}$, $A^{(7)}$, &c. in integral numbers.

SCHOLIUM.

In like manner may the law of continuation of the series for the secant be determined. Let

$$\text{secant } x = 1 + \frac{A^{(2)}}{1 \cdot 2} \cdot x^2 + \frac{A^{(4)}}{1 \cdot 2 \cdot 3 \cdot 4} \cdot x^4 + \frac{A^{(6)}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot x^6 + \&c.$$

then

$$A^{(2)} = 1$$

$$A^{(4)} = \frac{4 \cdot 3}{1 \cdot 2} \cdot A^{(2)} - 1$$

$$A^{(6)} = \frac{6 \cdot 5}{1 \cdot 2} \cdot A^{(4)} - \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot A^{(2)} + 1.$$

And

And in general

$$A^{(2n)} = \frac{2n \cdot 2n - 1}{1 \cdot 2} \cdot A^{(2n-2)} - \frac{2n \cdot 2n - 1 \cdot 2n - 2 \cdot 2n - 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot A^{(2n-4)} \\ + \frac{2n \cdot 2n - 1 \cdot 2n - 2 \cdot 2n - 3 \cdot 2n - 4 \cdot 2n - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot A^{(2n-6)} - \&c.$$

If we denote by M the meridional parts in Mercator's chart corresponding to the arch x : then

$$M = \int \frac{\dot{z}}{\cos x} = x + \frac{A^{(2)}}{1 \cdot 2} \cdot \frac{x^3}{3} + \frac{A^{(4)}}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{x^5}{5} + \&c. \text{ where} \\ A^{(2)}, A^{(4)}, A^{(6)} \&c. \text{ are as determined above.}$$

SECOND SOLUTION, communicated by PHILAETHES CANTABRIGIENSIS.

Many years have elapsed since an under-graduate of *Trinity College*, communicated to me, and to some other friends, the following method of computing the tangent of an arch in terms of the arch itself; which method is much easier than that which was used by Dr. Halley for the purpose about one hundred years ago, and by Mr. Emerson in his *Trigonometry* about forty years ago, (and by later writers,) viz. by dividing the sine by the cosine.

If the radius of a circle be called 1, and if t be put for the tangent of an arch denoted by z , then, $\frac{\dot{t}}{1 + tt} = \dot{z}$, and $z =$

$$t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}, \&c. \text{ which equations have often been given}$$

by the writers on fluxions. Now, from the first of these equa-

tions we have $\frac{\dot{t}}{z} = 1 + tt$; and, if we would express the value

of t in terms of z , we know from the second that it must be in series proceeding by the 1st, 3d, 5th, &c. powers of z . Let t be assumed $= Az + Bz^3 + Cz^5 + Dz^7 + Ez^9 + Fz^{11} +$

$$Gz^{13} + Hz^{15}, \&c. \text{ then, } \frac{\dot{t}}{z} = A + 3Bz^2 + 5Cz^4 + 7Dz^6$$

$$+ 9Ez^8 + 11Fz^{10} + 13Gz^{12} + 15Hz^{14}, \&c. \text{ But, since}$$

$$\frac{\dot{t}}{z} = 1 + tt, \text{ and } 1 + tt \text{ is } =$$

$$\begin{aligned}
& 1 + AAz^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + 2AEz^{10} + 2AFz^{12} + 2AGz^{14}, & \&c. \\
& \quad + BBz^6 + 2BCz^8 + 2BDz^{10} + 2BEz^{12} + 2BFz^{14}, & \&c. \\
& \quad \quad + CCz^{10} + 2CDz^{12} + 2CEz^{14}, & \&c. \\
& \quad \quad \quad + DDz^{14}, & \&c.
\end{aligned}$$

we now have an equation for determining the values of the coefficients A, B, C, &c. viz.

$$\begin{aligned}
& A + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + 11Fz^{10} + 13Gz^{12} + 15Hz^{14}, & \&c. = \\
& 1 + AAz^2 + 2ABz^4 + 2ACz^6 + 2ADz^8 + 2AEz^{10} + 2AFz^{12} + 2AGz^{14}, & \&c. \} \\
& \quad + BBz^6 + 2BCz^8 + 2BDz^{10} + 2BEz^{12} + 2BFz^{14}, & \&c. \} \\
& \quad \quad + CCz^{10} + 2CDz^{12} + 2CEz^{14}, & \&c. \} \\
& \quad \quad \quad + DDz^{14}, & \&c. \}
\end{aligned}$$

Hence we have $A = 1$,

$$B = \frac{AA}{3},$$

$$C = \frac{2AB}{5},$$

$$D = \frac{2AC + BB}{7},$$

$$E = \frac{2AD + 2BC}{9},$$

$$F = \frac{2AE + 2BD + CC}{11},$$

$$G = \frac{2AF + 2BE + 2CD}{13},$$

$$H = \frac{2AG + 2BF + 2CE + DD}{15},$$

&c.

where the law of continuation is evident.

But, since A is evidently $= 1$, $B = \frac{1}{3}$, $C = \frac{2}{15}$, and $D = \frac{17}{115}$, the above equations may be expressed in simpler terms, as here below:

$$A = 1,$$

$$B = \frac{A}{3},$$

$$C = \frac{2B}{5},$$

$$D = \frac{2C + \frac{1}{3}B}{7},$$

$$E = \frac{2D + \frac{2}{3}C}{9},$$

$$F = \frac{2E + \frac{2}{3}D + \frac{2}{15}C}{11},$$

$$G = \frac{2F + \frac{2}{3}E + \frac{1}{15}D}{13},$$

$$H = \frac{2G + \frac{2}{3}F + \frac{1}{15}E + \frac{1}{315}D}{15},$$

&c. &c.

where the law of continuation is again very obvious.

COROLL. Since it appears above, that $1 + tt$, the square of the secant of the arch z , is $= 1 + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + 11Fz^{10} + 13Gz^{12} + 15Hz^{14}$, &c. we have only to extract the square root of this series (of which the coefficients are now known), and we shall have the value of the secant. This may be done as follows:

Assume $\sqrt{(1 + tt)} = 1 + \zeta z^2 + \gamma z^4 + \delta z^6 + \varepsilon z^8 + \zeta z^{10} + \eta z^{12} + \theta z^{14}$, &c. then, by involution and substitution, we shall have

$$\begin{aligned} 1 + 3Bz^2 + 5Cz^4 + 7Dz^6 + 9Ez^8 + 11Fz^{10} + 13Gz^{12} + 15Hz^{14}, &c. = \\ 1 + 2\zeta z^2 + 2\gamma z^4 + 2\delta z^6 + 2\varepsilon z^8 + 2\zeta z^{10} + 2\eta z^{12} + 2\theta z^{14}, &c. \left. \begin{aligned} &+ 6\zeta z^4 + 26\gamma z^6 + 26\delta z^8 + 26\varepsilon z^{10} + 26\zeta z^{12} + 26\eta z^{14}, &c. \\ &+ \gamma\gamma z^8 + 2\gamma\delta z^{10} + 2\gamma\varepsilon z^{12} + 2\gamma\zeta z^{14}, &c. \\ &+ \delta\delta z^{12} + 2\delta\varepsilon z^{14}, &c. \end{aligned} \right\} \end{aligned}$$

from which equation,

$$\zeta = \frac{3B}{2},$$

$$\gamma = \frac{5C^2 - 6\zeta}{2},$$

$$\delta = \frac{7D - 26\gamma}{2},$$

$$\varepsilon = \frac{9E - 26\delta - \gamma\gamma}{2},$$

$$\zeta = \frac{11F - 26\varepsilon - 2\gamma\delta}{2},$$

$$\eta = \frac{13G - 26\zeta - 2\gamma\varepsilon - \delta\delta}{2},$$

$$\theta = \frac{15H - 26\eta - 2\gamma\zeta - 2\delta\varepsilon}{2},$$

&c. &c.

Here also the law of continuation is evident; and the expressions may be simplified by writing for ζ, γ , &c. their numerical values as fast as they arise.

XXIII. QUESTION 23, by Mr. JAMES CUNLIFFE,
Bolton.

To find the least biquadrate number, that can be divided into four integral parts, such, that the sum of every two of them may be a square number.

SOLUTION, by Mr. J. I.

Let n^4 denote the biquadratic number sought, and a, b, c, d , its four integral parts; then the question requires

$$\begin{aligned} a + b &= \mu^2 \\ a + c &= \mu'^2 \\ a + d &= \mu''^2 \\ c + d &= \nu^2 \\ b + d &= \nu'^2 \\ b + c &= \nu''^2 \end{aligned}$$

whence we get

$$n^4 = a + b + c + d = \mu^2 + \nu^2 = \mu'^2 + \nu'^2 = \mu''^2 + \nu''^2.$$

We see, therefore, that the biquadratic number sought must be resolvable into three squares three different ways; and, this condition being fulfilled, we derive

$$\begin{aligned} a &= \frac{1}{2}(\mu^2 + \mu'^2 - \nu''^2) \\ b &= \frac{1}{2}(\mu^2 + \nu'^2 - \mu'^2) \\ c &= \frac{1}{2}(\mu'^2 + \nu'^2 - \mu^2) \\ d &= n^4 - \frac{1}{2}(\mu^2 + \mu'^2 + \nu'^2), \end{aligned}$$

from which formulas we infer that the three squares, μ^2, μ'^2, ν'^2 , must either be all even, or two of them must be odd and only one even, in order that no fractions may arise from the divisions by 2; and likewise that the sum of every two of them must be greater than the third, and at the same time the sum of all the three less than $2n^4$, in order to avoid negative numbers.

It is, then, necessary to find a biquadratic number that may be divided into two squares three different ways. We owe to Fermat the curious discovery that every prime number exceeding a multiple of four by unity, or comprehended in the form $4x + 1$, is the sum of two squares. And this property, which is universal for prime numbers of the form $4x + 1$, belongs exclusively to that form; so that no prime number, which is not of

the form $4x + 1$, can possibly be the sum of two squares. This curious proposition was first demonstrated by the celebrated Euler; and to effect the demonstration it was necessary to prove this other proposition, viz. That every divisor of a number which is the sum of two squares that are prime to one another, is likewise the sum of two squares. From these principles we easily gather that the biquadratic number n^4 , we are seeking, must have prime divisors of the form $4x + 1$; for if it had none it could not possibly be the sum of two squares. And as we want a least value of n^4 , we may set aside all other divisors; for if n^4 had any other divisors, these could only enter, as common measures, into every two squares that compose n^4 . We may, therefore, suppose n^4 , and consequently n , to have no divisors but prime numbers of the form $4x + 1$.

And in the first place let us see whether a prime number p of the form $4x + 1$ will answer for n . We have already mentioned that p is the sum of two squares $\alpha^2 + \beta^2$; and we have now to add that p can be the sum of two squares only one way.

In like manner we shall find $p^2 = (\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2$; nor can p^2 be resolved into two squares in another way. Proceeding to the fourth power, we shall find,

$$\begin{aligned} p^4 &= (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)^2 + (4\alpha^3\beta - 4\alpha\beta^3)^2 \\ p^4 &= p^2\alpha^2 + p^2\beta^2. \end{aligned}$$

And these two are all the ways that p^4 can be resolved into two squares. Therefore, no single prime number of the form $4x + 1$ will answer the conditions required.

Let us now try the product of two prime numbers of the same form as p and p' . Then according to what has been shewn we shall have

$$\begin{aligned} p^4 &= (\alpha^4 - 6\alpha^2\beta^2 + \beta^4)^2 + (4\alpha^3\beta - 4\alpha\beta^3)^2 \\ p^4 &= p^2\alpha^2 + p^2\beta^2. \end{aligned}$$

And in like manner,

$$\begin{aligned} p'^4 &= (\alpha'^4 - 6\alpha'^2\beta'^2 + \beta'^4)^2 + (4\alpha'^3\beta' - 4\alpha'\beta'^3)^2 \\ p'^4 &= p'^2\alpha'^2 + p'^2\beta'^2. \end{aligned}$$

Therefore, combining the two values of p^4 with the two values of p'^4 , the biquadratic $p^4p'^4$ will be the product of the sums of two squares four different ways. But the product of two sums of two squares is the sum of two squares twice:

$$\begin{aligned} &\text{for } (M^2 + N^2) \cdot (P^2 + Q^2) \\ &= (MP + NQ)^2 + (MQ - NP)^2 \\ &= (MP - NQ)^2 + (MQ + NP)^2. \end{aligned}$$

Therefore,

Therefore, by combining the two values of p^4 with the two values of p'^4 , the biquadratic $p'^4 p^4$ will be the sum of two squares no less than eight different ways. To these eight we may add four other ways, by combining p^4 with the two values of p'^4 , and p'^4 with the two values of p^4 . Therefore, the fourth power of a number which is the product of two primes of the form $4x + 1$, is the sum of two squares, no less than twelve different ways.

We see then that no prime number of the form $4x + 1$, can satisfy the conditions required in the question. But a number which is the product of two such primes is the sum of two squares twelve different ways, one square of each sum being even, and one odd; therefore, as every combination of three out of the twelve, will give one solution of the question: and, as twelve things admit of 220 combinations of three, there will be so many different solutions, unless some are excluded as producing negative numbers. It is easy to see that one combination of three will give only one integral solution.

The two least primes of the form $4x + 1$ are 5 and 13, and their product 65 is consequently the least number that will satisfy the question. We have, $5 = 2^2 + 1$, and $13 = 3^2 + 2^2$: whence

$$65^4 = 3713^2 + 2016^2$$

$$65^4 = 2047^2 + 3696^2$$

$$65^4 = 2145^2 + 3640^2$$

which are only three of the twelve ways that 65^4 is the sum of two squares: And taking the three even squares for μ , μ' , and ν' , our formulæ give $a = 1826720$; $b = 2237536$; $c = 11422880$, and $d = 2363489$, which is one solution.

In like manner if we take another three of the twelve ways that 65^4 is the sum of two squares, as

$$65^4 = 2047^2 + 3696^2$$

$$65^4 = 2145^2 + 3640^2$$

$$65^4 = 4095^2 + 1040^2,$$

we shall derive another solution; $a = 746208$; $b = 22914208$; $c = 335392$; $d = 3854817$.

XXIV. QUESTION 24, by JACK CHANCE.

A in $a + b$ relations, tells b lies; and B in $c + d$ relations tells d of them false: What is the probability of the truth of a story in which they both agree?

SOLUTION,

SOLUTION, by the Proposer.

$(a + b) \times (c + d) = ac + bc + ad + bd =$ all the cases.

$ac =$ the number of times both agree in telling truth,

$bd =$ the number of times both agree in telling lies.

$bc =$ the times B tells truth, and A lies.

$ad =$ the times A tells truth, and B lies.

Therefore, $ac + bd =$ the whole number of times they both agree in telling both truth and lies.

But they are supposed to agree in their relation (by the question):

therefore, $\frac{ac}{ac + bd} =$ the probability required.

COROL. If the story were *untold*, the probability of their both telling true would be $\frac{ac}{ac + bc + ad + bd}$, that of their both lying would be $\frac{bd}{ac + bc + ad + bd}$, and that of their contradicting each other $\frac{bc + ad}{ac + bc + ad + bd}$.

Messrs. Bazley, Codling, Hearing, and Hopper, also sent Solutions to this question.

XXV. QUESTION 25, by QUINBUS FLESTRIN.

Given the base, the difference of the adjacent angles, and a line from the vertical angle to the middle of the base to construct the plane triangle.

FIRST SOLUTION, by Mr. JOHN WRIGHT, *Norley*.

CONSTRUCTION. Bisect the given base AB in O, (fig. 38, pl. 2.) with the right line ON, forming the angle NOB equal to the complement of half the given difference; and draw BQ perpendicular

perpendicular to ON. Constitute the right-angled triangle ONC, having its hypotenuse OC equal the given bisecting line, and such, that its area may be equal to that of the triangle OQB; then draw AC, BC, and ABC is the triangle required.

DEMONSTRATION. Because $OQ \times BQ = ON \times NC$, if the right line CI, terminating in AB, be drawn parallel to ON, it will bisect the angle ACB, as appears from Prob. 36, page 306 of Simpson's Algebra. Hence the difference of the angles ABC and BAC is equal to twice the complement of CIB or NOB, the given magnitude by construction.

Q. E. D.

SECOND SOLUTION, by Mr. JOHN DAWES, Birmingham.

ANALYSIS. Let ABC (fig. 39, pl. 2.) be the required triangle. Circumscribe it with the circle AMCB, and draw CM parallel to the base AB; continue CE to G, unite M, G; and M, E; and because EG is given (Euc. III. 35.); and ME and the angle MGC = the difference of the angles at the base are given by the question; therefore, the triangle MGE is given; consequently the triangle ABC is given. Hence this

CONSTRUCTION. Upon the give line CE (as radius) continued, and centre E describe the semicircle NMC and take EG so that GEC may be equal to the square of half the base; make the angle EGM equal half the given difference of the angles at the base, circumscribe the triangle CGM with the circle GMCB, and parallel to MC through E, draw AEB; then AC, CB being joined, ACB will be the triangle required.

Solutions were also received from Messrs. Butterworth, Collins, Farey, Johnson, Hopper, Morley, Myers, Pickering, and Scurr.

XXVI. QUESTION 26, by Mr. JAMES CUNLIFFE.

In a plane triangle there is given the perimeter, one of the angles, and the rectangle of the sides including that angle, to construct it.

FIRST

FIRST SOLUTION, by Mr. JOHN WRIGHT, *Norley.*

CONSTRUCTION. Fig. 40, pl. 2. Draw AB = the given perimeter, and make the angle $OAB = ABO =$ half the given one; with the centre O and radius OA or OB describe a circle, cutting BO produced in D ; divide DB in Q , so that BQ may have to DQ the ratio which the rectangle of the sides has to the square of BO , and concentric to the circle DAB , describe another through Q , cutting AB in I , through which point draw the diameter $COIH$, and having joined AH , BH , make the angles AHE , BHF , respectively equal to BAH , ABH , so shall HEF be the triangle required.

DEMONSTRATION. The angle $HEF + HFE = 2HAB + 2HBA = HOB + HOA = AOB$; consequently, $EHF = 2OAB =$ the given quantity; and EH being $= EA$, and $FH = FB$, $EH + FH + EF = AB$, the given perimeter. Moreover because the angle $HEI = BOI$, the triangles EHI , BOI , are equiangular, as also are the triangles FHI , AOI , for a similar reason.

Hence $EH \cdot HF : AO \cdot BO :: HI^2 : IB \cdot IA (HI \cdot IC) :: HI : IC$; that is, $EH \cdot HF : BO^2 :: BQ : DQ$.

Q. E. D.

SCHOLIUM. If the circle QII falls wholly below AB , the thing is impossible.

SECOND SOLUTION, by Mr. BAZLEY.

ANALYSIS. Suppose ACB (fig. 41, pl. 2.) the triangle required, C the given angle; then since the rectangle of the sides $AC \times CB$ is given as well as the angle C , the area of the triangle is given; but the area of the triangle is equal to half the rectangle under the perimeter and the radius of the inscribed circle, and since the perimeter is given, the radius of the inscribed circle is also given. Now, if O be the centre of the inscribed circle, and COL be drawn to meet the circumscribing circle in L , and OP be let fall perpendicular to CB ; we shall have the right-angled triangle COP wholly given, because the angle OCP is equal to half the given one, and OP being the radius of the inscribed circle; therefore, CO is given. But the rectangle $AC \times CB$ is given, and is well known to be equal to $OC \times (2LO + OC)$; but OC is given, consequently, LO and LC are given. Whence this

CONSTRUCTION.

CONSTRUCTION. Make $\angle ACB$ equal to the given angle, which bisect with the given line CL , on L as a centre, and with the radius LO describe a circle cutting CA , CB in A and B ; join AB , and $\triangle ACB$ is the required triangle, as is plain from the Analysis, and page 7, Vol. II. of the Old Series of the Repository.

*And thus the Question was answered by Messrs. Butterworth;
Cavill, and Farey.*

THIRD SOLUTION, by Mr. J. COLLINS, *Kensington.*

ANALYSIS. Let $\triangle ACB$ (fig. 42, pl. 2.) be the required triangle, $\angle ACB$ the given angle which bisect with the indefinite right line CFW . In CA produced take $CP =$ half the given perimeter, make PW perpendicular to AC , and let it meet CW in W ; then the right-angled triangle CPW is wholly given, and CS is known, because $CS \times (CF + SF) = CS \times SW = AC \times CB$ the given rectangle; therefore, all the parts of the triangle are known, and the construction evident.

*Thus the Question was answered by Messrs. Dawes and
Hopper.*

FOURTH SOLUTION, by Mr. MORLEY, *London.*

Draw any two lines CP , CP' (fig. 42, pl. 2.) to contain the given angle, and take CP , CP' each equal to half the perimeter, and erect the perpendiculars PW , $P'W$ meeting each other at W , and join CW which is known to bisect the vertical angle $\angle ACB$. In CW find the point S , so that $CS \times SW =$ the given rectangle, bisect SW in F and with the radius FS and centre F describe the circle $ASBW$, cutting AC , CB in A and B ; then AB being joined, $\triangle ACB$ will be the triangle required.

Mr. Johnson also answered this Question.

XXVII. QUESTION 27, by Mr. THOMAS BAZLEY,
Bolton.

In any plane triangle whereof C is the vertex, O the centre of the inscribed circle, Q the centre of the circle touching the base and sides produced, D the point where the line bisecting the vertical angle cuts the base, and S the point where the said line cuts the circumscribing circle below the base. Then the points being joined it is $QD : CD :: SD : OD$; required the demonstration.

FIRST SOLUTION, by Mr. JOHN FAREY.

Let the triangle ABC (fig. 43, pl. 2.) be circumscribed by the circle ACBS, let its vertical angle C be bisected by the line CQ, cutting the base in D, and the circumscribing circle in S; on S as a centre describe a circle to pass through A, and cut CQ in O and Q. Then, by Mr. Lowry's 1st. Prop. of the Repository, Vol. II. page 7, Old Series, O is the centre of the inscribed circle above the base, and Q the centre of the inscribed circle below the base; the circumference also passes through B the other extremity of the base. Now, since the circles ACBS and AOBQ cut each other, and ADB joins their points of intersection; we have,

by Euc. III. 35. $CD \times SD = AD \times BD$, also $OD \times QD = AD \times BD$;
Hence $CD \times SD = OD \times QD$, or $QD : CD :: SD : OD$.

Q. E. D.

SECOND SOLUTION, by Mr. JOHN BUTTERWORTH.

Let ABC (fig. 44, pl. 2.) be any plane triangle circumscribed by the circle ACBS; join the points as directed in the question; join also the points A, S; draw OE to make the angle OEC equal to the angle ODB. Then it is well known that the points C, O, D, S, Q, are in the same straight line CQ, and that $QS = SA = SO$; and the angle OEC being equal to the angle ODB, that is equal to the angle ADS, and the angle DCB equal to the angle SAD, the triangles COE and

and SAD are similar; hence, $AS (QS) : SD :: CO : OE$, and by composition $QS + SD = QD : SD :: CO + OE : OE$, but OE is equal to OD, because the angles OEC and ODB are equal; therefore, $QD : SD :: CD : OD$, or $QD : CD :: SD : OD$.

Q. E. D.

THIRD SOLUTION, *by Mr. JOHN DAWES.*

Let ABC be the triangle (fig. 45, pl. 2.). Because C, O, D, S, Q, are in a right line, perpendiculars from those points to the base AB will form similar triangles.

By hypothesis $QD : CD :: SD : OD$, theref. $HD : DE :: DK : DL$; but, $DK = DL + LK$ and $HD = HE + ED = EL + ED$; therefore, $ED + EL : ED :: DL + LK : DL$, or $EL : ED :: LK : LD$; therefore, $ED : EL :: LD : LK$, which is a well known property.

FOURTH SOLUTION, *by Mr. THOS. HOPPER, Manchester.*

Let ABC (fig. 45, pl. 2.) be the triangle circumscribed by a circle whose diameter SF is perpendicular to the base AB; C, O, D, S, and Q, the points mentioned in the question. Join SB and from Q upon AB, and CA produced let fall the perpendiculars QH, QI, which (as is well known) are equal to each other; also $SB = SO$ and the angle $SBE = QCI$. Wherefore by similar triangles,

$$IQ : IE :: CQ (= CD + QD) : BS (= SD + OD),$$

$$\text{and } HQ = IQ : SE :: QD : SD;$$

$$\text{consequently, } CD + QD : SD + OD :: QD : SD, \text{ or}$$

$$CD + QD : QD :: SD + OD : SD; \text{ and by division} \\ CD : QD :: OD : SD; \text{ whence, } QD : CD :: SD : OD.$$

Q. E. D.

According to one or other of these methods were the Solutions given by Messrs. Bazley, Cavill, Collins, Johnson, and Morley.

XXVIII. QUESTION 28, *by Mr. BAZLEY.*

Given the vertical angle, the radius of the inscribed circle, and the sum, difference, or ratio of the rectangle of the two sides including the vertical angle and the rectangle of the two lines drawn from the extremes of the base to the centre of the inscribed circle; to construct the plane triangle.

SOLUTION, *by Mr. JOHN CAVILL, Brighton.*

ANALYSIS. Suppose ACB (fig. 46, pl. 2.) the triangle required, O the centre of the inscribed circle through which COS is drawn meeting the circumscribing circle below the base in S; now it is well known that a circle described about the point S with the radius SA or SB will pass through O and intersect the greater side AC in b, making Cb = CB. Let R and P be the points of contact of the radius of the inscribed circle, with the base AB and the side CB; and join OR, OP. Then, because the angle PCO = half the given vertical angle, and the angle P right, and since OP the radius of the inscribed circle is given, the hypothenuse OC is given. Again, by a well known theorem $2SO \times OR = AO \times OB$, and by what is done above, $AC \times CB = AC \times Cb =$ (by Euc. Cor. to 36. III.) $OC \times (2SO + OC)$. Hence, when the sum or difference of the rectangles is given, $(2SO + OC) \times OC + 2SO \times OR$ is given; that is, $2SO \times OC + OC^2 + 2SO \times OR = 2SO \times (OC + OR) + OC^2$ is given, but since OR and OC are given, SO is given.

Lastly, when the ratio of the rectangles is given we shall have (by taking $AC \times CB : AO \times OB :: n : m$ in the given ratio) $n : m :: 2SO \times OC + OC^2 : 2SO \times OR$; therefore, since CO and OR are given, SO is given. Wherefore, SO being found, all the cases of the construction may be as follows.

CONSTRUCTION. Having drawn the lines CA, CB to include the given angle and having also drawn the given inscribed circle to touch them in P and p; through O the centre draw COS making OS of the given length found in the Analysis, and on S as a centre with the radius SO describe the circle AOB cutting AC and CB in A and B respectively; join AB and ABC will be the triangle required.

And thus the Question was answered by Messrs. Bazley, Butterworth, Collins, Dawes, Johnson, and Morley.

XXIX. QUESTION 29, *by Mr. BAZLEY.*

There is an hemispherical bowl, whose diameter at the brim is 20 inches, placed with its axis perpendicular to the horizon; an uniformly heavy thin inflexible rod or right line, whose length is 24 inches and weight 8 ounces, is sustained in equilibrio by resting on the edge of the brim at some distance from the upper extremity, whilst the lower end rests against the concave surface of the bowl. It is required to determine by a general investigation, the position of the rod and the respective pressures at the two points of support.

FIRST SOLUTION, *by AMICUS.*

Let ARE (fig. 47, pl. 2.) be a section of the bowl and rod perpendicular to the horizon; O the centre of the bowl, and AOE an horizontal diameter; let G be the middle of the rod DR, or its centre of gravity, and draw GH perpendicular to AE and OK perpendicular to DR.

Put $OE = r = 10$; $RG = a = 12$; and the part of the rod within the bowl, $RE = x$; then, $EK = \frac{x}{2}$, and OK

$= \sqrt{(r^2 - \frac{x^2}{4})}$: but $OE : OK :: EG : GH$, that is, $r :$

$$\sqrt{(r^2 - \frac{x^2}{4})} :: x - a : GH = \frac{1}{r} (x - a) \sqrt{(r^2 - \frac{x^2}{4})}.$$

In whatever point of view the Theory of Statics has been considered, the following proposition has either been assumed as a self-evident maxim, or it has been shewn to be a necessary consequence of the first principles, on which every Theory is built; viz. That a system of heavy bodies will be in equilibrium when the common centre of gravity has reached the point of lowest descent, or of highest ascent; or to speak more mathematically, when the common centre of gravity is in that particular point of the locus which it describes, when it passes from descending to ascending, or ascending to descending. But although the system of bodies will be equally in equilibrium when the common centre of gravity exactly occupies the point of lowest descent, or of highest ascent, yet there will be a great difference in the states of equilibrium in the two cases. For in the one case, when the centre of gravity is in the point of lowest descent, or when it passes from descending

ing to ascending, the equilibrium will have stability; and although the centre of gravity be displaced a little from the exact situation it should occupy, it will have a tendency to regain its place, and will oscillate about the point of equilibrium. On the other hand, when the centre of gravity is in the point of highest ascent, or when it passes from ascending to descending, the equilibrium will be unstable; and if the system be moved so as to displace the centre of gravity from its precise place ever so little, there will be no tendency to regain the situation of equilibrium; but, on the contrary, the system will remove more and more from it.

In the present problem it is manifest that the centre of gravity G will descend, and the situation of the rod, where G is the lowest, will be the equilibrium required. Now G will be in the lowest place, when GH is a maximum, that is, when its fluxion,

making x variable, is $= 0$. Therefore, $(r^2 - \frac{x^2}{4}) - \frac{x}{4}(x-a)$

$$= 0; \text{ whence, } x^2 - \frac{a}{2}x = 2r^2, \text{ and } x = \sqrt{2r^2 + \frac{a^2}{16}} + \frac{a}{4}.$$

In the particular case proposed $x = \sqrt{209} + 3 = 17.45$, &c.

To determine the pressures at E and R draw RL perpendicular to AE , and let it meet OK produced in S . Then the forces by which the rod is sustained in equilibrium are these, viz. the gravity of the rod acting in the direction SR ; the pressure on the brim of the bowl at E acting perpendicularly to the rod, or in the direction KS ; and lastly, the pressure at R acting in the direction RO , perpendicularly to the bottom of the bowl: and the proportions of the forces will be the same as the proportions of the sides of the triangle SOR formed by lines parallel to directions in which the forces act; or, the forces will be as the sines of the angles of the triangle SOR , that is, as the lines RK , KO , OL . Hence, putting w for the weight of the rod, the pressure on the brim

$$\text{of the bowl} = \frac{OK}{RK} \times w = \frac{\sqrt{r^2 - \frac{x^2}{4}}}{\frac{1}{2}x} \times w = 4.47 \text{ oz.}$$

$$\text{And the pressure on the bottom} = \frac{OL}{RK} \times w = \left(\frac{x}{r} - \frac{2r}{x}\right) \times w = 4.8 \text{ ounces.}$$

SECOND SOLUTION, by the Proposer.

Let $ABCOD$ (fig. 48, pl. 2.) be a section of the bowl and rod passing through the axis CB , and the rod $ROGD$, G the middle

middle of the rod, or its centre of gravity; from O draw OS perpendicular to RD, and from D through the centre C draw DCS to meet OS in S, from S draw SIG, meeting ACO in I; join AD, and draw GT parallel to DS, meeting SO in T. Then by the nature of the Question, the re-action of the edge of the bowl at O is in the direction OS, and the re-action of the concavity at D is in the direction DCS; therefore, by Emerson's Mechanics, prop. 53. 4to Edit. the line SIG passing through the rod's centre of gravity is perpendicular to the horizon; whence, SI is perpendicular to ACO. This being premised, put AO the diameter of the bowl = d , GD or GR = a , and DO = x . Then, AD = SO = $\sqrt{(d^2 - x^2)}$, and by the similar triangles ADO, GOS, it is OD : AD :: AD = OS :

$$OG, \text{ that is, } x : \sqrt{(d^2 - x^2)} :: \sqrt{(d^2 - x^2)} : \frac{d^2 - x^2}{x} = GO,$$

$$\text{whence, } DG = a = x - \frac{d^2 - x^2}{x}; \text{ therefore, } 2x^2 - d^2 = ax,$$

or $x^2 - \frac{1}{2}ax = \frac{1}{2}d^2$, and consequently, $x = \sqrt{(\frac{1}{2}d^2 + \frac{1}{16}a^2)} + \frac{1}{4}a$, by which the position of the rod is determined, and in the case proposed, where $d = 20$, $a = 12$, we get $x = OD = 17.45683$ inches.

Again, by the Principles of Mechanics, the rod is sustained by three forces, whose quantities and directions are as SG, GT, TS, respectively; that is, the weight of the rod, the re-action of the concavity at D, and the re-action of the brim at O. Hence putting the weight of the rod = w , by the similar triangles SGT, SDG, we get

$$SG : GT :: SD : SG :: d : \frac{d}{x}\sqrt{(d^2 - x^2)} :: w : \frac{\sqrt{(d^2 - x^2)}}{x} \times w$$

= the pressure in the direction SD = (in the present case where w 8 oz. and $x = 17.45683$) 4.472784 ounces.

Also, SG : ST :: SD : DG :: $d : a :: w : \frac{a}{d} \times w$ = the pressure in the direction SO = 4.8 oz. in the present case.

COROLLARY. If $a = d$, the expression for x becomes = d ; and in this case the expression for the pressure against the concavity becomes = 0, whilst that for the pressure on the brim becomes = w . Again, when a vanishes as is the case when a weight = w is connected to the lower extremity of the rod, (the rod itself being considered to have no weight) we have $x = \frac{1}{2}d$, which shews that the point D then coincides with B the lowest point in the bowl; and in this case the expression for the pressure against the concavity becomes = w , whilst that for the brim vanishes, as it ought. Moreover it

evidently

evidently appears that a must not exceed d , as otherwise the rod would fall out of the bowl.

THIRD SOLUTION, by Mr. H. CODLING, *South Walsham.*

Suppose BIH (fig. 49, pl. 2.) a perpendicular section of the bowl through the centre C, CI the axis, RD the rod in the required position, G its centre of gravity (or middle of the rod), RZ, LGM and BK parallel, and RK perpendicular to the axis CI. Then, in order to simplify the problem, let us suppose the weight of the rod to be collected in the point G, for such supposition will make no difference with regard to the pressures on R and B.

Now, if the weight of the rod be represented by BR, it is evident, by the resolution of forces, that it will be sustained by two forces RK and BK, the one acting perpendicular and the other parallel to the axis of the vessel; then, if the total pressure be represented by RK, that exerted at R will evidently be as RL, and that at B as the sum of the forces BK, KL, or by composition as BL; but these forces cannot be known till the position of the rod is determined, which may be done as follows: It is a known principle in Mechanics, that the centre of gravity will descend to the lowest place it can get when the rod rests in equilibrio; therefore, MG must be a maximum; then making $HZ = v$, $BH = d$, $GR = r$, by sim. Δs , $HR^2 = HB \cdot HZ = dv$, and $RB = \sqrt{(d^2 - dv)}$, therefore, $GB = \sqrt{(d^2 - dv)} - r$, and $d : \sqrt{(dv)} :: \sqrt{(d^2 - dv)} - r : GM = \frac{\sqrt{(d^3v - d^2v^2)} - r\sqrt{(dv)}}{d} = a \text{ maximum};$

this being put into fluxions and reduced gives $v = \frac{1}{2}d - \frac{r^2 + r\sqrt{(r^2 + 8d^2)}}{8d} = 4.76295$; consequently, $BL = HR =$

$\sqrt{(dv)} = 9.760072$, $RL = d - 2v = 10.474099$, and $BR = 17.45683$. Now putting w the weight of the rod = 8 oz. we

have the pressure on R = $\frac{BL}{RB} \times w = \sqrt{\left(\frac{dv}{d^2 - dv}\right)} \times w =$

4.47274 ounces and the pressure at B = $\frac{RL}{RB} \times w =$

$\frac{d - 2v}{\sqrt{(d^2 - dv)}} \times w = 4.8 \text{ ounces.}$

XXX. PRIZE QUESTION 30, by PHILALETHER
CANTABRIGIENSIS.

Given $\frac{6xx^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = y$, in which
equation x and y begin together to find the value of y when x
becomes $= 10$.

FIRST SOLUTION, by the Proposer, PHILALETHER
CANTABRIGIENSIS.

The denominator of the given fraction is the product of the
three trinomial factors, $xx+3x+3$, $xx-2x+2$, and $xx-x+1$,
which may be discovered by Sir Isaac Newton's method of di-
visors, or more easily by the method explained by Madam
Agnesi, in her *Analytical Institutions*, Vol. I. p. 160. We may
therefore, resolve the given fraction into three fractions, by the
method explained in p. 138, of Vol. II. of the work before

mentioned, thus: Assume $\frac{6xx^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6}$
 $= \frac{Ax\dot{x} + D\dot{x}}{xx + 3x + 3} + \frac{Bx\dot{x} + E\dot{x}}{xx - 2x + 2} + \frac{Cx\dot{x} + F\dot{x}}{xx - x + 1}$, (omitting the
coefficient 6, in the numerator of the given fraction, which will
a little shorten the calculation) and find the values of A , B , C ,
&c. which will be as follow: viz.

$$A = \frac{180}{7 \times 61}, \quad D = \frac{459}{14 \times 61},$$

$$B = \frac{44}{61}, \quad E = \frac{48}{61},$$

$$C = -\frac{1}{7}, \quad \text{and} \quad F = -\frac{3}{14}.$$

Now, the fraction

$$\begin{aligned} \frac{Ax\dot{x} + D\dot{x}}{xx + 3x + 3} &\text{ is evidently } = \frac{Ax\dot{x} + \frac{2}{3}A\dot{x}}{xx + 3x + 3} + \frac{(D - \frac{2}{3}A)\dot{x}}{xx + 3x + 3}, \\ \frac{Bx\dot{x} + E\dot{x}}{xx - 2x + 2} &\dots\dots\dots = \frac{Bx\dot{x} - B\dot{x}}{xx - 2x + 2} + \frac{(E + B)\dot{x}}{xx - 2x + 2}, \\ \frac{Cx\dot{x} + F\dot{x}}{xx - x + 1} &\dots\dots\dots = \frac{Cx\dot{x} - \frac{1}{2}C\dot{x}}{xx - x + 1} + \frac{(F + \frac{1}{2}C)\dot{x}}{xx - x + 1}; \end{aligned}$$

the correct fluents of which, are,

$$1^{\text{st}}. \quad \frac{1}{2} A \times \text{h. l.} \left(\frac{xx}{3} + x + 1 \right),$$

$$2^{\text{d}}. \quad + \frac{1}{2} B \times \text{h. l.} \left(\frac{xx}{2} - x + 1 \right),$$

$$3^{\text{d}}. \quad + \frac{1}{2} C \times \text{h. l.} (xx - x + 1),$$

$$4^{\text{th}}. \quad + \frac{1}{3} (D - \frac{1}{2} A) \times \left\{ \begin{array}{l} \text{circ. arch of tang. } x + \frac{3}{2} \\ - \text{circ. arch of tang. } \frac{3}{2} \end{array} \right\} \text{ and radius } \frac{\sqrt{3}}{2},$$

$$5^{\text{th}}. \quad + (E + B) \times \left\{ \begin{array}{l} \text{circ. arch of tang. } x - 1 \\ + \text{circ. arch of tang. } 1 \end{array} \right\} \text{ and radius } 1,$$

$$6^{\text{th}}. \quad + \frac{1}{3} (F + \frac{1}{2} C) \times \left\{ \begin{array}{l} \text{circ. arch of tang. } x - \frac{1}{2} \\ + \text{circ. arch of tang. } \frac{1}{2} \end{array} \right\} \text{ and radius } \frac{\sqrt{3}}{2}.$$

And the values of these fluents, (according to my calculation,) when x becomes $= 10$, are as below : viz.

+	—
1 st . = 0.799195,	3 ^d . = 0.322204,
2 ^d . = 1.339322,	4 th . = 0.049113,
6 th . = 0.330490.	5 th . = 0.147248;
+ 2.469007	— 0.518565
— 0.518565	

$$\frac{1.950442}{6} = \frac{y}{6},$$

and $y = 11.702652$, which was required.

SECOND SOLUTION, by Mr. I. H. HEARDING, *Adderbury*.

It is easy to discover, by Sir Isaac Newton's method of divisors, or still more easily by *Agnesi's Analytical Institutions*, Vol. I. p. 160, that the denominator of the given fraction is the product of $(x^2 + 3x + 3) \times (x^2 - x + 1) \times (x^2 - 2x + 2)$; therefore, by the above-mentioned *Institutions*, Vol. II. page 138,

138, we may assume $\frac{x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} =$
 $\frac{Ax + P}{x^3 + 3x + 3} + \frac{Bx + Q}{x^3 - x + 1} + \frac{Cx + R}{x^3 - 2x + 2}$, (the quantity
 $6x$ being here purposely omitted to shorten the calculation).
 Then, by reducing these fractions to a common denominator,
 and equating the coefficients of the homologous terms, we shall
 have $A+B+C-1=0$, $-3A+B+2C+P+Q+R=9$,
 $5A-B+C-2P+Q+2R=0$, $-4A+5P-Q+R=0$,
 $2A+6B+3C-4P=0$, and $2P+6Q+3R=0$; and
 from these equations A is found $= \frac{180}{7 \cdot 61}$, $B = -\frac{1}{7}$, $C = \frac{44}{61}$,

$P = \frac{459}{14 \cdot 61}$, $Q = \frac{3}{14}$, and $R = -\frac{48}{61}$. Now, by substituting $v - \frac{2}{3}$, $u + \frac{1}{2}$, and $w + 1$, respectively for x , in the three foregoing fractions, in order to exterminate the second terms of the denominators, and then multiplying the first side of the resulting equation by $\dot{x}$, the first fraction on the last side by $\dot{v}$, the second by $\dot{u}$, and the third by $\dot{w}$, there will at length arise

$$\frac{\dot{x}x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = \frac{A\dot{v} + S\dot{v}}{vv + \frac{1}{2}} + \frac{B\dot{u} + T\dot{u}}{uu + \frac{1}{2}} + \frac{C\dot{w} + V\dot{w}}{ww + 1},$$

where $-\frac{2}{3}A + P = -\frac{81}{14 \cdot 61}$ is put $= S$, $\frac{1}{2}B + Q = \frac{1}{7} = T$,

and $C + R = -\frac{4}{61} = V$.

The three fractions on the right hand side of the last equation are evidently equal to the sum of these six following, viz.

$$= \frac{A\dot{v}\dot{v}}{vv + \frac{1}{2}} + \frac{S\dot{v}}{vv + \frac{1}{2}} + \frac{B\dot{u}\dot{u}}{uu + \frac{1}{2}} + \frac{T\dot{u}}{uu + \frac{1}{2}} + \frac{C\dot{w}\dot{w}}{ww + 1} + \frac{V\dot{w}}{ww + 1};$$

the fluents of which gives the fluent of

$$\frac{\dot{x}x^5}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} =$$

$$\frac{1}{2}A \times \text{hyp. log. } (vv + \frac{1}{2}) + \frac{1}{2}B \times \text{h. l. } (uu + \frac{1}{2}) + \frac{1}{2}C \times \text{h. l. } (ww + 1)$$

$$+ \frac{4S}{3} \times \text{circular arc to tangent } v \text{ and radius } \frac{1}{2}\sqrt{3},$$

$$+ \frac{4T}{8} \times \text{circular arc to tangent } u \text{ and radius } \frac{1}{2}\sqrt{3},$$

$$+ V \times \text{circular arc to tangent } w \text{ and radius } 1.$$

(K 2)

And,

And, by restoring the values of v , u , and w , this last expression becomes =

$$\frac{1}{2} \times h.l. (x^2 + 3x + 3) + \frac{1}{2} B \times h.l. (x^2 - x + 1) + \frac{1}{2} C \times h.l. (x^2 - 2x + 2)$$

$$+ \frac{4S}{3} \times \text{circular arc to tangent } x + \frac{1}{2} \text{ and radius } \frac{1}{2}\sqrt{3},$$

$$+ \frac{4T}{3} \times \text{circular arc to tangent } x - \frac{1}{2} \text{ and radius } \frac{1}{2}\sqrt{3},$$

$$+ V \times \text{circular arc to tangent } x - 1 \text{ and radius } 1.$$

But as this should vanish when $x = 0$, the fluent of

$$\frac{xx^3}{x^6 - x^4 + 2x^3 + 5x^2 - 6x + 6} = \frac{y}{6} \text{ when properly corrected is}$$

$$\frac{1}{2} A \times h.l. (\frac{1}{3}x^2 + x + 1) + \frac{1}{2} B \times h.l. (x^2 - x + 1) + \frac{1}{2} C \times h.l. (\frac{1}{2}x^2 - x^2 + 1)$$

$$\left. \begin{array}{l} + \frac{4S}{3} \\ + \frac{4T}{3} \\ + \frac{4}{V} \end{array} \right\} \times \text{circular arc to tangent} \left\{ \begin{array}{l} x + \frac{1}{2} \\ x - \frac{1}{2} \\ x - \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{l} \text{minus the} \\ \text{arc whose} \\ \text{tangent} = \end{array} \right\} \left\{ \begin{array}{l} \frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{array} \right\} \left\{ \begin{array}{l} \text{and} \\ \text{common} \\ \text{radius} = \end{array} \right\} \left\{ \begin{array}{l} \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} \\ 1. \end{array} \right\}$$

$$= 1.9504408$$

$$\text{And, consequently, } y = 1.9504408 \times 6 = 11.7026448.$$

This Question was also answered by Messrs. Johnson, Mabbott, and Wright.

PHILAETHES CANTABRIGIENSIS is requested to send to Mr. GLENDINNING's for the Medal for Solving the Prize Question.

ERRATA.

In Question 51, No. I. For, "is $\frac{1}{2}\sqrt{(h^2 + r^2)} + \frac{1}{2}S + \frac{1}{2}s$:"
Read, "is a mean proportional between $\frac{1}{2}\sqrt{(h^2 + r^2)} + \frac{1}{2}S + \frac{1}{2}s$ and r ."

NOTICES

RELATING TO MATHEMATICS.

I. REGULAR POLYGON OF SEVENTEEN SIDES.

Mr. GAUSS, of Brunswiek, published at Leipfic, in 1801, a work called *Disquisitiones Arithmeticae*. In this work the author announces that we may always inscribe a regular polygon of $2^n + 1$ sides in a circle, when n is a whole number, and $2^n + 1$ a prime number. Mr. LEGENDRE has given to the French National Institute a demonstration of this very curious proposition, in the case when the number of sides is 17. It is founded on these two lemmas.

LEMMA I. Let a be an arch of a circle, m and n two whole numbers, then

$$2 \cos. ma \cos. na \mp \cos. (m - n) a \pm \cos. (m + n) a.$$

LEMMA II. Let a be the n th part of the whole circumference, n being a whole number, then

$$\cos. a + \cos. 2a + \cos. 3a + \&c. \dots \dots + \cos. na = 0.$$

These two propositions are generally known.

By the second of them we have

$$\cos. a + \cos. 2a + \cos. 3a + \&c. \dots + \cos. 15a + \cos. 16a + \cos. 17a = 0.$$

and, because $\cos. 17a = 1$.

$$\cos. a + \cos. 2a + \cos. 3a + \&c. \dots + \cos. 15a + \cos. 16a = -1 \dots (E).$$

Now, if we remark that $\cos. a = \cos. 16a$, $\cos. 2a = \cos. 15a$, $\cos. 3a = \cos. 14a$, &c. it will immediately appear that equation (E) may be decomposed into two others (E') and (E'').

$$\left. \begin{aligned} &\cos. a + \cos. 3a + \cos. 5a + \cos. 7a \\ &+ \cos. 9a + \cos. 11a + \cos. 13a + \cos. 15a \end{aligned} \right\} = -\frac{1}{2} \dots (E')$$

$$\left. \begin{aligned} &\cos. 2a + \cos. 4a + \cos. 6a + \cos. 8a \\ &+ \cos. 10a + \cos. 12a + \cos. 14a + \cos. 16a \end{aligned} \right\} = -\frac{1}{2} \dots (E'')$$

Again,

Again, equation (E') may be resolved into these two,

$$\left. \begin{array}{l} \text{cof. } 3a + \text{cof. } 5a + \text{cof. } 7a + \text{cof. } 11a = m \\ \text{cof. } a + \text{cof. } 9a + \text{cof. } 13a + \text{cof. } 15a = n \end{array} \right\} \dots\dots (c')$$

where m and n are two such quantities that $m + n = -\frac{1}{2}$ and $mn = -1$.

This second resolution may be verified by taking the product of the equations, (c') and changing, by the first lemma, the products of the cosines of the arches into the cosines of their sum and difference.

Again each of the equations (c') may be decomposed into two others by making $m = p + q$, $n = r + s$, thus,

$$\left. \begin{array}{l} \text{cof. } 3a + \text{cof. } 5a = p \quad \text{cof. } a + \text{cof. } 13a = r \\ \text{cof. } 7a + \text{cof. } 11a = q \quad \text{cof. } 9a + \text{cof. } 15a = s \end{array} \right\} \dots\dots (f)$$

Here, $pq = rs = \frac{1}{4}$; the truth of these equations may be verified as before.

We have now obtained six equations

$$\begin{array}{lll} m + n = -\frac{1}{2} & p + q = m & r + s = n \\ mn = -1 & pq = -\frac{1}{4} & rs = -\frac{1}{4} \end{array}$$

and from these the values of m, n, p, q, r, s , may be found by the resolution of equations, each of the second degree.

The four equations (f) may, by lemma 1, be written thus,

$$\begin{array}{ll} \text{cof. } 3a + \text{cof. } 5a = p & \text{cof. } 6a + \text{cof. } 7a = q \\ \text{cof. } 3a \text{ cof. } 5a = \frac{s}{2} & \text{cof. } 6a \text{ cof. } 7a = \frac{r}{2} \\ \text{cof. } a + \text{cof. } 4a = r & \text{cof. } 2a + \text{cof. } 9a = s \\ \text{cof. } a \text{ cof. } 4a = \frac{p}{2} & \text{cof. } 2a \text{ cof. } 9a = \frac{q}{2} \end{array}$$

these eight cosines, viz. $\text{cof. } a, \text{cof. } 2a, \text{cof. } 3a, \text{cof. } 4a, \text{cof. } 5a, \text{cof. } 6a, \text{cof. } 7a, \text{cof. } 9a$, may be found by the resolution of equations of the second degree. The remaining eight, viz. $\text{cof. } 16a, \text{cof. } 15a, \text{cof. } 14a, \text{cof. } 13a, \text{cof. } 12a, \text{cof. } 11a, \text{cof. } 10a, \text{cof. } 8a$, are respectively equal to the former eight, as has been already observed. Thus it appears that the description of a regular polygon of 17 sides is a problem which may be effected by means of straight lines and circles; that is, by the methods of construction employed in the elements of plane geometry.

II. ORBITS OF THE NEW PLANETS.

The elements of the orbits of the two new Planets have now been determined with tolerable accuracy: they are thus given by La Lande*.

* From the *Journal de Physique*, Brumaire, an. 12.

Piazzi, or Ceres, discovered January 1, 1801.

Revolution 4 years, 7 months, 10 days.			
Mean longitude, January 1, 1804	105	11° 59'	
Annual motion	2	18	14
Aphelion	10	26	44
Node	2	21	6
Equation of the orbit	0	9	3
Excentricity		0.097	
Inclination	0	10	37

Olbers, or Pallas, discovered March 28, 1802.

Revolution 4 years, 7 months, 11 days.			
Mean longitude, January 1, 1804	95	29° 53'	
Annual motion	2	18	11
Aphelion	10	1	7
Node	5	22	28
Equation of the orbit	0	28	25
Excentricity		0.2463	
Inclination	0	34	39

III. FIGURE OF THE EARTH.

The length of a degree of the meridian, measured in Lapland by Maupertuis, le Monnier, &c. agreed so ill with theory, and with the other measurements, as to give good grounds to suspect considerable error. The Swedish Astronomer Melanderhielm has caused all the operations of that measurement to be carefully repeated: and the result of this labour differs from the old determination no less than 196 toises: the length of a degree of the meridian in latitude $66^{\circ} 20'$ coming out 57,209 toises instead of 57,405 according to the French Academicians. The new result agrees better with the various measurements in different latitudes.

The French Astronomer Mechain is employed in continuing to the Balearian Islands the measurement of the meridian in France, conducted by him and Delambre.

In the East Indies, Major Lambton, who is employed by the Presidency of Madras to survey the Mysore country, has already measured an arch of the meridian $1^{\circ} 34' 56'' 43$ in extent, which gives the degree in latitude $12^{\circ} 32'$ equal to 60,494 fathoms.

IV. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philosophical Transactions of the Royal Society of London. Abridged by Charles Hutton, L. L. D. F. R. S. George Shaw, M. D. F. R. S. F. L. S. Richard Pearson, M. D. F. A. S.

A Course of Mathematics, designed for the use of the Officers and Cadets of the Royal Military College. Vol. I. By Isaac Dalby, Professor of Mathematics in the said College.

Traacts on the Resolution of Cubic and Biquadratic Equations. By Francis Maferes, Esq. F. R. S. Curstitor Baron of the Erchequer.

A Geometrical Treatise of Conic Sections, in Four Books, to which is added a Treatise on the Primary Properties of Conchoids, the Cissoid, the Quadratrix, Cycloids, the Logarithmic Curve, and the Logarithmic, Archimedian, and Hyperbolic Spirals. By the Rev. Abram Robertson, A. M. F. R. S. Savillian Professor of Geometry in the University of Oxford.

An Experimental Enquiry into the Nature and Propagation of Heat. By John Leslie.

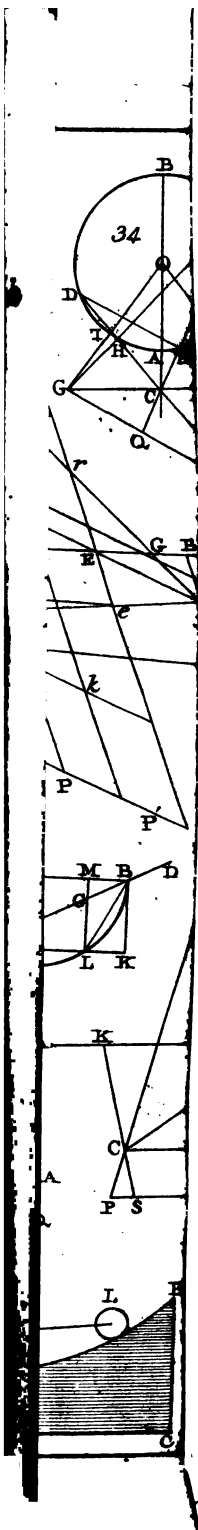
Recreations in Mathematics and Natural Philosophy; containing Dissertations concerning the most proper Subjects to excite curiosity and attention to Mathematical and Philosophical Sciences. The whole treated in an easy manner, and adapted to the comprehension of all who are in the least initiated in the Sciences of Arithmetic, Geography, Navigation, Architecture, Chemistry, &c. &c. First composed by M. Ozanam, and now translated into English, and improved, with many Additions. By Charles Hutton, L. L. D. F. R. S.

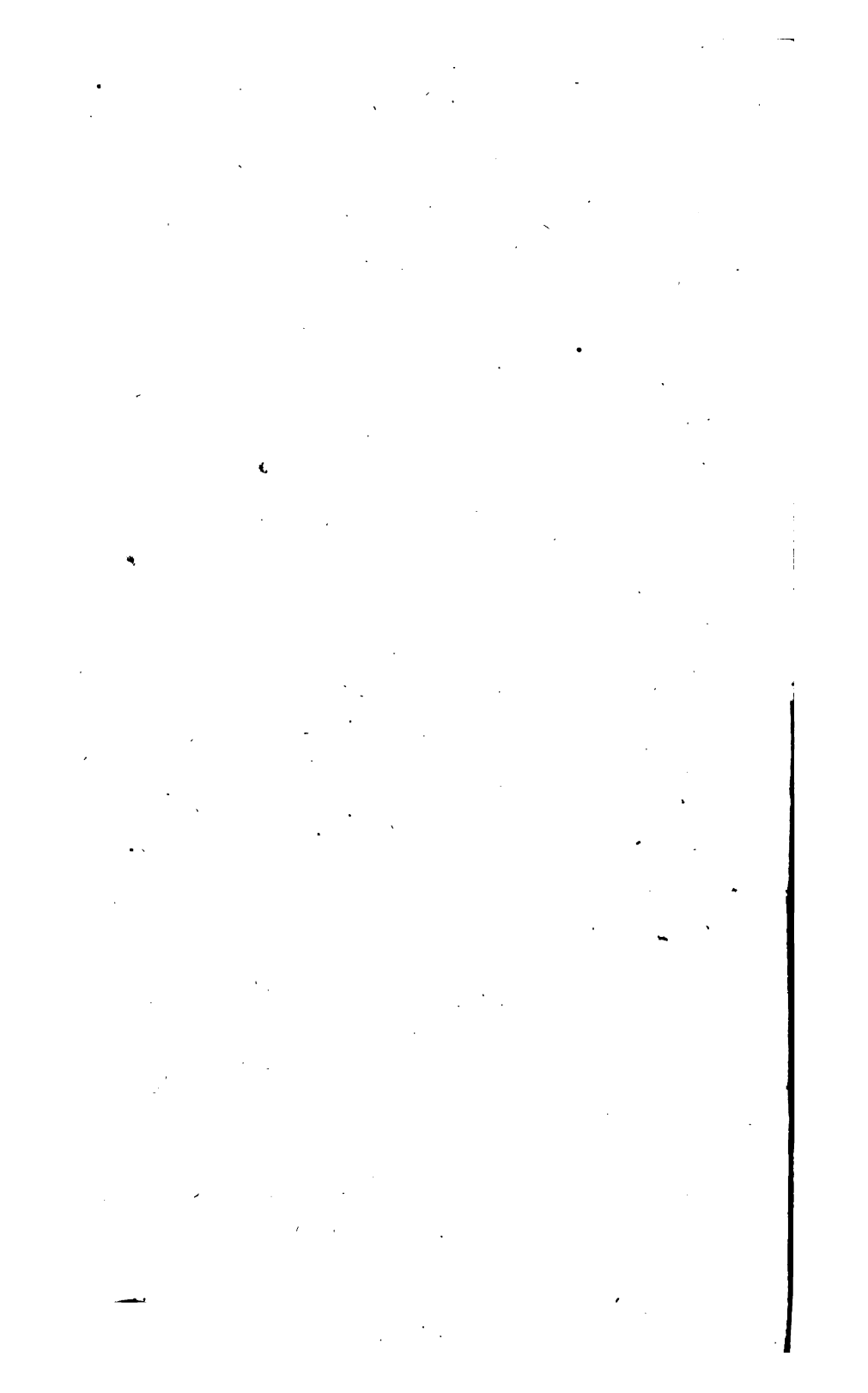
A General History of Mathematics, from the earliest times to the middle of the Eighteenth Century. Translated from the French of John Bosjut. To which is affixed, a Chronological Table of the most eminent Mathematicians. By John Bonnycastle.

MATHEMATICAL WORKS IN THE PRESS.

Professor Robinson, of Edinburgh, has in the Press a *Work on the Elements of Mechanical Philosophy*, which will contain the substance of the excellent Lectures he has for many years read in the University of Edinburgh.

A New Edition of Professor Playfair's Elements of Geometry is soon to appear.





ARTICLE II.

Solutions to Questions proposed in Number I.

I. QUESTION 31, by Mr. T. MILNER.

Being out with a quadrant, some of my pupils were desirous of knowing the height of the parish church spire, which rises to a considerable height in the middle of the town of Sheffield; to gratify their curiosity I found two stations in Church Street, distant from each other 70 yards, exactly on a level with the bottom of the spire, such that the altitude of the top of the spire at each station was equal to $35\frac{1}{2}$ degrees; and again, in the middle of this station line, the altitude of the spire's top was found to be $36\frac{1}{2}$ degrees. From hence it is required to find geometrically the height of the said spire, and its distance from the last station, supposing the quadrant to be held 5 feet from the ground at each station.

FIRST SOLUTION, by Mr. THOMAS HOPPER, Manchester.

CONSTRUCTION. On any given line AB (fig. 52, pl. 3.) taken at pleasure, describe the two right-angled triangles BCA, BDA, making the angle $\angle BCA = 36\frac{1}{2}^\circ$ and $\angle BDA = 35\frac{1}{2}^\circ$. Draw CF at right angles to AC and equal to 35 yards, and thereto apply $AE = AD$, let AE produced meet FG, drawn parallel to AC, in G; draw GH parallel to FC meeting AC produced in H; and draw HI parallel to CB meeting AB in I. Then will G and H be two stations and AI the height of the spire.

DEMONSTRATION. Join EB, then it is evident from the construction that the points D, A, C, E, F, G, and H, are in the same horizontal plane. Now, because HI is parallel to CB, the $\angle AHI = \angle ACB = 36\frac{1}{2}^\circ$, the angle subtended by the spire at the middle station H. Again, by sim. $\triangle$ s, $AB : AI :: AC : AH :: AE : AG :: EB : GI$; therefore GI is parallel to EB, and therefore $\angle AEB = \angle AGI$; but the right-angled triangles EAB, ABD, having BA common, and $AD = AE$ are equal in every respect; hence the $\angle AGI = \angle ADB = 35\frac{1}{2}^\circ$, the angle subtended

FIRST SOLUTION, by Mr. THOS. MYERS, *Hammer Smith.*

It is manifest, from a little consideration, that this question is no more than to find what annuity payable half-yearly for 25 years may be purchased for 500*l.* allowing compound interest at the rate of 5 per cent. Then, from the Theorem on page 168 of Simpson's *Dissertations*, we have the half-yearly payment =

$$\frac{R-1}{1-\frac{1}{R^n}} \times A = 17*l.* 12*s.* 6*d.* 3\frac{1}{4}\text{qrs.}; \text{ where } A = 500,$$

$R = 1.025$, and $n = 50$, the number of payments.

The same result will also be obtained if we find the amount of 500*l.* at 5 per cent. per ann. compound interest for 25 years, and then find what annuity, payable half-yearly, for the same time, and at the same rate per cent. will be equal to the former amount.

SECOND SOLUTION, by Mr. HOPFER.

In this case 500*l.* may be considered as the present worth of an annuity to continue 25 years; which annuity may be found by Theo. 2, page 238, Simpson's *Algebra*. Thus, by using the same notation, we have $R = 1.025$, $n = 50$, $V = 500$; and

$$\log. A = \log. V + \log. (R - 1) - \log. \left(1 - \frac{1}{R^n}\right) = 17*l.*$$

12*s.* 7*d.* nearly, the required annuity.

THIRD SOLUTION, by Mr. GEORGE PICKERING.

Let x be the sum to be deducted at the end of each half-year, r = the ratio, and $a = 500$; then

$ar - x$ is the sum remaining after the 1st half-year,
 $ar^2 - xr - x$ 2^d half-year,
 $ar^3 - xr^2 - xr - x$ 3^d half-year,
 &c. till

ar

$ar^{50} - xr^{49} - xr^{48} \dots \dots \dots xr - x \dots \dots \dots$ 50th half-year.
that is, $ar^{50} - x \times : 1 + r + r^2 + r^3 \dots \dots \dots + r^{49}$ is equal
to the sum remaining after the 50th half-year; but the sum of
the geometrical series

$1 + r + r^2 \dots \dots \dots + r^{48} + r^{49}$ is $\frac{r^{50} - 1}{r - 1}$, and

because the whole sum was to be received at the end of the 50th
half-year, therefore, the above expression must be equal to no-

thing, that is, $ar^{50} - x \times \frac{r^{50} - 1}{r - 1} = 0$, or $x \times \frac{r^{50} - 1}{r - 1} =$

ar^{50} ; consequently $x = \frac{ar^{50} \times (r - 1)}{r^{50} - 1} = \frac{ar^{51} - ar^{50}}{r^{50} - 1} =$

17l. 12s. 7d. nearly.

*Other Solutions were given by Messrs. Cavill, Collins, Johnson,
Marratt, and Merones Minor.*

III. QUESTION 33, by Mr. I. H. HEARDING, *Adderbury.*

Given $\left\{ \begin{array}{l} x + \frac{x}{1.3} + \frac{x}{2.3} + \frac{x}{3.5} + \frac{x}{3.7} + \&c. \text{ ad infin.} = 30y \\ \frac{1}{3} + y + 2y^2 + \frac{1}{3}y^3 + 5y^4 + \&c. \text{ ad infin.} = \frac{2}{10}x \end{array} \right.$
to find x and y .

SOLUTION, by Mr. WM. MARRATT, *Sibsey, near Boston.*

By Simpson's Algebra, page 217, the sum of the infinite series

$x + \frac{x}{1.3} + \frac{x}{2.3} \&c.$ is $=$ to $2x$.

And, multiplying the second equation by 3 we get,

$1 + 3y + 6y^2 + 10y^3 + \&c. = \frac{27x}{10}$; that is, by Simpson's

Algebra, page 219, $\frac{1}{(1-y)^3} = \frac{27x}{10}$ or $\frac{1}{3(1-y)^3} = \frac{9x}{10}$.

Hence

Hence the equations given in the question become,

$$2x = 30y$$

$$\frac{9x}{10} = \frac{1}{3(1-y)^3}$$

Now, by writing $1 - z$ instead of y in these two equations, we have

$$x = 15 \times (1 - z)$$

$$x = \frac{10}{27z^3}$$

$$\text{That is, } 15 \times (1 - z) = \frac{10}{27z^3}$$

$$\text{or } z^4 - z^3 + \frac{2}{81} = 0.$$

$$\text{Hence, } z = \frac{1}{3}, y = \frac{2}{3}, \text{ and } x = 10.$$

*Nearly in the same manner was the Answer given by
Mr. Harding, the Proposer.*

IV. QUESTION 34, by Mr. W. FRANCIS, Taplow.

A circular piece of ground, containing 220 acres, is to be divided between three persons, A, B, C, whose shares are to be as 5, 6, 7. Query the shares, by construction, so that the boundaries may be equal?

FIRST SOLUTION, by Mr. GEORGE PICKERING.

The diameter of the circle is readily found $= 131.417$ yards. Now let this be divided into three parts, AB, BC, CD, (fig. 55, pl. 3.) having the same proportion as the numbers 5, 6, and 7; and on AB, AC, CD, and BD, describe semicircles, then the curvilinear spaces *aa*, *bb*, *cc*, are the parts required, as is evident from Dr. Hutton's 8th Tract, where this curious division of circles into parts having equal perimeters and given ratios was first noticed. The content of each share is easily found, for $5 + 6 + 7 = 18$, and as 18 is to 220, so is each of the numbers 5, 6, 7, to the respective shares, viz. $61\frac{1}{3}$, $73\frac{1}{3}$, and $85\frac{2}{3}$.

*And thus it was answered by Messrs. Collins, Hopper,
Johnson, and Myers.*

SECOND

SECOND SOLUTION, *by Mr. JOHN CAVILL, Beighton.*

Let PQR (fig. 56, pl. 3.) be the circular piece of ground, the centre of which is O; divide the periphery into 18 equal parts, then take $PQ = 5$, $RP = 6$, and $QR = 7$ of those parts; take also $Qd =$ one of those parts, and with the centre d and radius dO describe the arc $Ob = \text{arc } dQ$; again, take $Rr =$ one half of one of the equal parts, and with the centre r and radius rO describe the arc $Oa = \text{arc } Rr$; join ar and bd cutting RO in n and m respectively. Then will $PObd = A$'s share, $raObd = C$'s share, and $POar = B$'s share. For the circle is evidently divided in the proportion required by the radii OP , OQ , OR , and the spaces mOb , nOa are respectively equal to the spaces mQd and nrR ; and $PQ + Qd + Ob = dr + aO + bO = Pr + aO + Rr =$ seven parts; the radius being common to each boundary.

V. QUESTION 35, *by MERLIN.*

To cut 5 equal and similar parts from 5 equal squares, so that the 10 pieces when joined together shall form a square.

SOLUTION, *by MERLIN, Jun.*

If 1 be the side of each of the 5 equal squares, then, into whatever figures they may be cut, the side of the greater square must be $\sqrt{5}$.

Let ABCD be one of the 5 squares: bisect AD in G; then if the square is cut through the line GC it will be divided into a right-angled triangle GDC, and a trapezoid AGCB; and if 4 other like squares are cut in the same manner, and the parts placed round the square ABCD, (as in fig. 57, pl. 3.) the whole together will form the square EROP. This is too evident from the diagram itself to need a formal demonstration.

Suppose $AD = 1$, then $\sqrt{1\frac{1}{4}} = GC = RS = SO$, and $2\sqrt{1\frac{1}{4}} = \sqrt{5} = RO$.

Nearly in the same manner was the answer given by Messrs. Boole, Cavill, Collins, Dawes, Hopper, Myers, and Pickering,

VI. QUES.

VI. QUESTION 36, by Mr. I. T. M'DONELD.

Given the sum of the base and one of the sides, and the excess of the base above the other side, to construct the plane triangle when its area is either a given space or a maximum.

SOLUTION, by Mr. JOHN CAVILL, *Beighton*.

Suppose ACB (fig. 58, pl. 3.) to be the triangle required; put $AB + BC = s$, $AB - BC = d$, $BC = x$, $AC = AB = y$, and the given area $= a^2$; then, by a well known theorem,

$a^2 = \frac{1}{4} \sqrt{(s^2 - x^2)(x^2 - y^2)}$ or $16a^4 = (s^2 - x^2)(x^2 - y^2)$. Now $AC = s - AB$, and $AB = d + x$, therefore, $AC = s - d - x$, and $AC - AB = s - d - 2x = y$; this value of y being substituted in the above equation, and then properly reduced, we obtain

$$3x^4 - 12(s - 2d)x^3 + 2(4d^2 - 4sd - 2s^2)x^2 + s^2(4s - 2d)x + (s - 2d)^2s^2 \Big\} = \frac{a^4}{16}.$$

From which equation x may be readily determined when the given quantities are expressed in numbers.

Again, when the area is a maximum, the fluxion of the above equation must be made $= 0$, and then we have

$$x^3 - (s - 2d)x^2 + \frac{1}{6}(4d^2 - 4sd - 2s^2)x = \frac{1}{12}s^2(4s - 2d).$$

Hence it appears that the problem is not capable of a geometrical construction even in the extreme case.

VII. QUESTION 37, by Mr. JAMES WHALLEY, *Bolton*.

Having a circle and a triangle ABC, both given in magnitude and position; it is required to draw a tangent to the circle, cutting BC in P and AB in Q, so that drawing PR parallel to AB, the sum, difference, or ratio of the two triangles PQB and PRC may be given.

SOLUTION,

SOLUTION, *by* MERONES MINOR.

Suppose the line PQK drawn as required, touching the given circle, whose centre is O at K; draw DOG parallel to BA, meeting CB produced in G, and PK in D, and on GO demit the perpendicular BL; then by similar triangles, $BP : PQ :: GB : BI$, and $OD : OK :: BI : BL$; therefore, by compounding, $BP \cdot OD : PQ \cdot OK :: GB : BL$. Take m a fourth proportional to BL , GB and OK , and then $BP \cdot OD = m \cdot PQ$.

Again, by similar triangles, $BP : BQ :: GB : GI$, or $BQ \cdot GB = BP \cdot GI$; therefore, $BP \cdot IO = BP \cdot (GO - GI) = BP \cdot GO - BQ \cdot GB$, or $BP \cdot (IO + OD) = BP \cdot BQ = BP \cdot GO - BQ \cdot GB + m \cdot PQ$; that is, $PG \cdot QB - BP \cdot GO = m \cdot PQ$, or $(PG \cdot QB - BP \cdot GO)^2 = m^2 \cdot PQ^2$. Now, putting v = the versed sine of the angle QBP, $a = BC$, $b = BG$, $c = GO$, $x = BP$, and $y = BQ$, we have by trigonometry, $PQ^2 = (x - y)^2 + 2vxy$, and the above becomes

$$((b + x)y - cx)^2 = m^2 ((x - y)^2 + 2vxy) \dots \dots E.$$

Again, because the angle PBQ is given, the triangle QBP has a given ratio to the rectangle $BQ \cdot BP$; and the triangle RPC being given in species it will have a given ratio to PC ; therefore, when the triangles QBP, RPC, have a given ratio, the rectangle $QB \cdot BP$ will have to PC^2 a given ratio, which suppose to be that of p to q ; then $PC = a - x$, and $xy : (a - x)^2 :: p : q$; now from this equation, and the preceding one, we may obtain the values of x and y by an equation of the 6th power.

Again, when the sum or difference of the triangles QBP, RPC is given, the sum or difference of the rectangle $QB \cdot BP$ and a space to which PC^2 has a given ratio is given. Suppose this space to be $= n \cdot PC^2$, and that the given sum or difference is $= s^2$; then $xy \pm n(a - x)^2 = s^2$, and from this equation and the equation E, the values of x and y may be found by an equation of the same dimensions as in the former case.

VIII. QUESTION 38, *by* Mr. WILLIAM SMITH, *Liverpool*.

Let AB be the axis of a conic section, from B draw BD to meet the curve in D, demit DC perpendicular to AB, and produce it from C till CP is in a given ratio to BD; the locus of the point P will be a conic section. Required the demonstration.

FIRST SOLUTION, by Mr. SMITH, the Proposer.

Fig. 60, Pl. 3. Produce PC to H so that $BC \cdot CH = CD^2$, and draw AHE to meet BE (drawn to make half a right angle with AB) in E, then because $AC \cdot CB$ has to CD^2 ($BC \cdot CH$) the same ratio as the square of AB to the square of the other axis, the ratio of AC to CH will be given, and, therefore, AE will be given by position, and BE is given by position, therefore, if EF be drawn perpendicular to AB, BF will be of a given length. Now when AB is the greater axis, AC will be greater than CH, and the point E must be below AB. In this case $BD^2 = (BC^2 + CD^2 = CQ^2 + CQ \cdot CH) = CQ \cdot QH$, therefore, $CP^2 : CQ \cdot QH$ in a given ratio, but CQ is to CB, and QH to FC, always in a given ratio, therefore, $CP^2 : BC \cdot CF$ is a given ratio, and, therefore, the locus of the point P is a hyperbola.

Again, when AB is the less axis, then AC is less than CH, and the point E will be above AB, and by reasoning as above, we have $CP^2 : FC \cdot CB$ in a given ratio, and, therefore, in this case the locus of P is an ellipse, when the given conic section is an ellipse, but a hyperbola as before when the given section is a hyperbola.

Again, when the section is a parabola, HE (fig. 61, pl. 3) is parallel to AB, and CH is equal to the parameter of the given parabola, and in this case $BC \cdot CF$ has to CP^2 a given ratio, and the locus of P is a hyperbola.

SECOND SOLUTION, by Mr. THOMAS BAZLEY, Bolton.

I. For the ellipsis (fig. 62, pl. 3.) Put the axis $AB = a$, and its conjugate $2OE' = b$, $BC = x$, and the ratio of $BD : CP ::$

$d : a$. Then, per conics, $a^2 : b^2 :: ax - x^2 : CD^2 = \frac{b^2}{a^2} (ax$

$- x^2)$, consequently, $DB^2 = \frac{b^2 ax}{a^2} - \frac{b^2}{a^2} x^2 + x^2 = \frac{b^2 ax}{a^2}$

$+ \frac{a^2 - b^2}{a^2} x^2$, and, therefore, $CP^2 = \frac{a^2}{d^2} \times BD^2 =$

$\frac{b^2 ax + (a^2 - b^2) x^2}{a^2} = \frac{a^2 - b^2}{a^2} \left(\frac{b^2 a}{a^2 - b^2} x - x^2 \right)$, which

(when a is greater than b) is an equation to the hyperbola,
whose

whose axes are $\frac{b^2}{a^2 - b^2} \times a$ and $\frac{b^2}{d \sqrt{(a^2 - b^2)}} \times a$. But if b is the greater, the equation becomes $CP^2 = \frac{b^2 ax - (b^2 - a^2)x^2}{d^2} = \frac{b^2 - a^2}{d^2} (\frac{b^2 a}{b^2 - a^2} x - x^2)$; an equation to the ellipsis, whose axes are $\frac{b^2}{b^2 - a^2} \times a$ and $\frac{b^2}{d \sqrt{(b^2 - a^2)}}$. Moreover, if $b = a$, the ellipsis becomes a circle, and the equation for the value of CP becomes $CP^2 = \frac{b^2 ax}{d^2}$; an equation to the parabola, whose parameter is $\frac{b^2}{d^2} \times a$.

II. For the hyperbola, (fig. 63, pl. 3.); the same notation remaining, $DC^2 = \frac{b^2}{a^2} ax + \frac{b^2}{a^2} x^2$, by the property of the curve; consequently $CP^2 = \frac{a^2}{d^2} \times BD^2 = \frac{b^2}{d^2} ax + \frac{a^2 + b^2}{d^2} \times x^2 = \frac{a^2 + b^2}{d^2} (\frac{b^2 ax}{a^2 + b^2} + x^2)$, which expresses an hyperbola, whose axes are $\frac{b^2}{a^2 + b^2} \times a$, and $\frac{b^2}{d \sqrt{(a^2 + b^2)}} \times a$.

III. For the parabola, (fig. 64, pl. 3.). Put the parameter $= p$, then $CD^2 = px$, and $BD^2 = px + x^2$; therefore $CP^2 = \frac{a^2}{d^2} (px + x^2)$, the equation to an hyperbola whose axes are p , and $\frac{a}{d} \times p$.

THIRD SOLUTION, by MERONES MINOR.

First, let the given conic section be a parabola whose parameter is BE , (fig. 64, pl. 3.) and take $BF : BE$ in the given ratio of $CP : BD$; then, by the property of the parabola, $BC \cdot BE = DC^2$; and, therefore, $DB^2 = BC \cdot BE + BC^2 = BC \cdot CE$; wherefore, $BE^2 : BF^2 :: BC \cdot CE : CP^2$, and the locus of the point P is a hyperbola, whose vertex is B , and axes BE and BF .

When the given ratio is that of equality, the axes are equal, and in that case the hyperbola is equilateral.

(M 2)

Again,

Again, let the given conic section be an ellipse, as in (fig. 62, pl. 3.) or an hyperbola, as in (fig. 63, pl. 3.) and BI the respective parameters; then, $AB : BI :: AC \cdot BC : DC^2$, or $\frac{BI}{AB}$

$\times AC \cdot BC \doteq DC^2$, and, therefore, $DB^2 = \frac{BI}{AB} \cdot (AB \mp BC)$.

$BC + BC^2 = \frac{BI}{BA} \cdot AB \cdot BC + \frac{AI}{AB} \cdot BC^2$; hence, taking BE

a fourth proportional to AI, BI, and AB, we have $AB : AI :: (BE + BC) \cdot BC = BC \cdot EC : DB^2$. Now, take $BE^2 : BH^2 :: AB : AI$, and $BF : BH$ in the given ratio of CP : DB; then, $BE^2 : BH^2 :: BC \cdot EC : DB^2$, and $BF^2 : BH^2 :: CP^2 : DB^2$; therefore, $BE^2 : BF^2 :: BC \cdot CE : CP^2$, and consequently the locus of P is a hyperbola, whose axes are BE and BF, and vertex B. This is general for the hyperbola, whether AB be the greater or less axis; but in the ellipse when AB is the less axis it will appear in a manner similar to the above, that the locus will be an ellipse.

Answers were also received from Messrs. Hopper and Johnson.

IX, QUESTION 39, by Mr. THOMAS BAZLEY, Bolton.

From the given point P in the periphery of a given semi-circle APR, it is required to draw the right line PB meeting the diameter AR in B, so that BC being drawn perpendicular to AR meeting the periphery in C; $BC^2 - BP^2$ shall be of a given magnitude.

FIRST SOLUTION, by Mr. J. JOHNSON, Birmingham.

CONSTRUCTION. On AR (fig. 65, pl. 3.) demit the perpendicular PD, divide OD at B, (O being the centre) so that $2OB \cdot BD$ may be equal to the given difference of the squares of BC and BP; draw BP and it is the line required.

Join OP, BC; then, $BC^2 = OC^2 - OB^2$, and $BP^2 = OC^2 (OP^2) - OB^2 - 2OB \cdot BI$ (Eu. 12. II.); therefore, $CB^2 - PB^2$ is $= 2OB \cdot BD$, the given difference. The difference is evidently a maximum when OI is bisected at B, and a minimum when B coincides with O.

SECOND

SECOND SOLUTION, *by Mr. BAZLEY, the Proposer.*

CONSTRUCTION. Fig. 65, Pl. 3. Let M^2 = the given magnitude. Then, from the centre O draw OS making the $\angle AOS = \frac{1}{2}$ a right $\angle$, and on AR drop the perpendicular PD; from D, with a radius equal to the side of the square ($OC^2 - PD^2 - M^2$), describe a circle cutting OS in G and g , draw GB, gb , perpendicular to AO, and join PB, Pb , and the thing is done, either of the points B, b , answering the conditions of the question.

DEMONSTRATION. Join OC and OP. Then, because $\angle BOG = \angle BGO$, BO will be = BG, and $DB^2 + BO^2 = DG^2 = OC^2 - PD^2 - M^2$ per confr. whence $M^2 = OC^2 - OB^2 - (PD^2 + DB^2) = BC^2 - PB^2$.

Q. E. D.

COR. If BC be equal to BP, then DG will be equal to DO, and the points B, b , will fall in O and D. Also, when M^2 is a maximum, GD^2 is evidently a minimum, and in that case DG is perpendicular to OS, and the points B, b , coincide in the middle of DO,

Ingenious Solutions to this question were also received from Messrs. Cavill, Hopper, and Merones Minor.

X. QUESTION 49, *by GEOMETRICUS.*

Let two circles have the same centre, and let A, B, be two given points in the circumference of one of them. It is required to inflect AC, BC, to the circumference of the other circle, so that the rectangle AC $\times$ CB may be a given magnitude.

FIRST SOLUTION, *by Mr. THOMAS BAZLEY.*

GEOMETRICAL ANALYSIS. Suppose the lines AC, CB, (fig. 66, pl. 3.) drawn as required, produce AC to meet the exterior circle

circle in D, and join BD. Through the centre O and the point C draw the diameter ECF. Then, since the circles are given and concentric, the rectangle $EC \cdot CF$ is given and constant. But $EC \cdot CF = AC \cdot CD$; therefore, $AC \cdot CD$ is given, and by hypothesis $AC \cdot CB$ is given; consequently the ratio of $AC \cdot CD : AC \cdot CB$ is given, that is, the ratio of $CD : CB$ is given. But since the points A, B are given, the angle ADB is given, and, therefore, the triangle CBD is given in kind; hence the angle BCD as well as its supplement ACB is given. Therefore, if we describe on AB the segment of a circle to contain the given angle ACB, it will meet the interior circle in the point or points required.

SECOND SOLUTION, by Mr. JOHNSON, Birmingham.

Through the centre O (fig. 67, pl. 3.) draw the diameter CEOH, meeting the smaller circle in E and H; also, let AC, BC, meet the same circle in D and F, and join AF; then because the given circles are concentric, CE and CH will be given in magnitude, and $CB \cdot CF =$ the given rectangle $CE \cdot CH$; but by the question the rectangle $CA \cdot CB$ is given, therefore, the ratio of AC to CF is given, and the angle AFC (the supplement of the given angle AFB) being given, the triangle ACF will be given in species, and consequently the angle ACB will be given; hence if a segment of a circle capable of containing this angle be described on AB, its intersection with the greater of the given circles will determine the point C.

The rectangle $AC \cdot CB$ will be a maximum, when $AC = CB$, if ABE be the greater segment, and a minimum when $Ac = cB$, but *vice versa* when ABE is the less segment. The solution will be exactly the same when A, B, are in the circumference of the greater circle.

XI. QUESTION 41, by HYPATIA.

Three points being given by position, it is required to construct a triangle, the sides of which may be given in magnitude, and pass through the given points.

SOLUTION,

SOLUTION, by Mr. THOMAS BAZLEY, Bolton.

Suppose it done, ABC (fig. 68, pl. 3.) the required Δ , D, E, F the given points. Let the points D, E, F be joined: then since the sides of the Δ ABC are given, the $\angle$ s are also given; and because DE, DF, and FE are given, the circles described on those lines to contain the given $\angle$ s, C, A, and B, are given in magnitude and position. Let DCE, DAF be two of these circles, and the question is then reduced to this:

To draw through D, the point of intersection, the right line ADC of a given length, terminated by their peripheries. For then the points A, C, being determined, we have only to draw AFB, CEB, through the points F and E, to meet in B, and the thing is done. Hence the following construction:

Let S, O, the centres of the circles DCE, DAF, be joined, and on OS describe a semicircle, in which apply $SG = \frac{1}{2}AC$, and through D draw CDA parallel to SG to cut the circles in C and A; draw CEB, AFB, meeting in B, and ABC is the Δ required.

DEMONSTRATION. Through O draw OGK to meet AC in K, and SI parallel to OK meeting AC in I; then because $\angle$ G is right, and GS, KI, as also GK, SI, parallels, therefore, GI is a rectangle, and SI, GK, are perpendicular to AC; consequently $KD = \frac{1}{2}AD$, and $DI = \frac{1}{2}DC$, therefore, $KD + DI = (KI = GS) \frac{1}{2}DA + \frac{1}{2}DC = \frac{1}{2}AC$: that is, $GS = \frac{1}{2}AC$, and the rest is plain from the analysis.

And thus the Question was answered by Messrs. Collins, Johnson, Lowry, Merones Minor, Myers, Pickering, and Smith.

XII. QUESTION 42, by GEOMETRICUS.

A triangle being given by position, and two points being also given, it is required to draw lines through these points to intersect each other in one of the sides of the triangle, and meet the other sides so as to cut off segments adjacent to given points in them, such that the rectangle contained by these segments may be equal to a given space.

FIRST SOLUTION, by Mr. LOWRY, R. M. College.

ANALYSIS. Let ABC (fig. 69, pl. 3.) be the triangle given by position, E and F the given points, through which lines are to be drawn to meet in one of the sides as AB, and H, L, the given points in the other sides AC, BC. Now, suppose the problem resolved, and that IFG, KEG, are drawn to intersect in AB, and meet AC, BC, in I and K, so that the rectangle HI·LK may be equal to a given space. Draw FD parallel to AC, meeting AB in D, and join DE; then DE will evidently be given by position. Through the points F, E, draw a line meeting AC in N, and BA produced in O, and draw IQ parallel to OE, meeting GE in Q. Let AQ joined meet GE in M, draw MP parallel to EG, and join NP.

Then, by reason of the parallels, $AP : PG :: AM : MQ :: AN : NI$; therefore, PN is parallel to GFI, and, consequently, $OP : PG :: ON : NF :: OM : ME :: OA : AD$; therefore, AM is parallel to DE; but, DE is given by position, and A is a given point, therefore, AMQ is given by position, and M will be a given point, and NM, ME, given lines. Draw EV making the angle MEV = to the given angle MAN, then EV will evidently be a given line; on OE produced take ET = EV, and draw TR, parallel to ED, and equal to HN; then, because, ET is a given line, and E a given point, T will also be a given point, and, therefore, TR, which is parallel to DE will be given by position, and also in magnitude, being equal to the given line HN, therefore, R is a given point. Now, join RE, and let RT meet GE in S; then, by similar triangles, $AM : AN :: ME : EV = ET :: MQ : TS$, and $AM : AN :: MQ : NI$, therefore, $TS = NI$, and NH is = to TR, therefore, RS is = HI, and, consequently, the rectangle RS·LK is = to the given rectangle HI·LK; the problem is, therefore, reduced to this; viz. to draw a straight line through E to intersect BC and TR in K and S, so that the rectangle of the segments LK, RS, adjacent to the given points L, R, may be equal to a given space, which is the same as the *Sectio Spatii* of the ancient geometricians. See Dr. Halley's Translation, and also the Solution to Q. 220, Old Series of the Repository.

It is obvious that the above analysis is applicable, when the points K, L, are in the side AC instead of BC, viz. when the problem requires IFG, KEG, to be inflected to AB, meeting AC in I and k, so that the rectangle IH·kl may be equal to a given space, H and l being given points in the line AC; which is a case that frequently occurs in the solution of problems.

SECOND

SECOND SOLUTION, by Mr. W. SMITH, *Liverpool.*

ANALYSIS. Let Aqs (fig. 70, pl. 3.) represent the given triangle, d, w the given points in the sides, and P, Q the other given points; and suppose QRt, PRT to be drawn as required to intersect Aq in R , and meet ds in t , and As in T , and make the rectangle $wT \times dt =$ to a given space. Join Qd , and draw QI parallel to ds meeting As in I , and IS parallel to Qd , meeting Qt in S , then I will be a given point, and IS will be given by position. By the similar triangles Qdt, QIS , we have $dt \times IS =$ the given rectangle $Qd \times QI$, but by the problem the rectangle $dt \times wT$ is given; therefore, the ratio of IS to wT is given. We have, therefore, to inflect two lines QRS, PRT , from the given points Q, P , to intersect Aq in R , and meet IS in S , and As in T respectively, so that IS may have to wT a given ratio, which reduces the problem to the same as question 17.

Mr. Johnson also answered it.

XIII. QUESTION 43, by SIDROPHEL.

If a triangle given in species have two of its angles upon straight lines given by position and the side adjacent to these angles passing through a given point; required the *Locus* of the angle opposite to that side.

FIRST SOLUTION, by Mr. W. SMITH, *Liverpool.*

ANALYSIS. Let LX, LY (fig. 71, pl. 3.) be the lines given in position, O the given point, and ABC the triangle given in species. Draw BD making the angle $BDC = \angle BLA$, then the ratio of AD to DC is given; also the locus of the point D is a right line given in position, as LZ , for the points L, A, D, B are in a circle, and, therefore, the angle $BLD (= \angle BAD)$ is given. Draw AF , making the angle $LAF = \angle BAD$, and from C draw CF parallel to LD meeting AF in F , then the angle AFC will be given, and if OG be

drawn to meet AL in G making the angle $\text{OGA} = \angle \text{AFC}$, then G will be a given point, and the triangles OAG, CAF will be similar; and, therefore, $\text{AG} : \text{OG} :: \text{AF} : \text{FC}$, or $\text{AG} \cdot \text{FC} = \text{OG} \cdot \text{AF}$. From G draw GkK, meeting LF in k, and CF in K, then if AF meet LD in f, the ratio of AD : DC being given, that of AF : fF will be given, and the locus of F will be a right line given in position, as LF; therefore, AG has to AF a given ratio; viz. that of GL to Lk, and AF has to LF a given ratio also, viz. that of Gk to kL, and OG being a given line, the rectangle kF·FC will be equal to the rectangle under LF and a given line, equal, suppose to LF·LR, and, therefore, the parallelogram kFCH = the parallelogram LFRr, or the parallelogram IrCH = the parallelogram LKIR, that is, the rectangle Ir·rC (= Lk·kI) is given; and, therefore, the locus of C will be a hyperbola, whose asymptotes are Ir and IH.

SECOND SOLUTION, by Mr. LOWRY, R. M. College.

Let ABC (fig. 72, pl. 3.) be the triangle given in species, having the angles A, B, on the straight lines EA, EB given by position, and the side AB passing through the given point P. Draw EH to make the angle BEH equal to the given angle BAC, then EH will be given by position. Conceive CH and BH to be drawn so as to make the angle CHB equal to the given angle BEA, and join BK, K being the point where CH meets AE; then, because, the angles KHB, KEB are equal, the points E, H, K, B are in a circle; therefore, the angles BKC, BEH, BAC are equal, and, consequently, the points B, K, A, C are also in a circle; and, therefore, the angles EKB, (EHB), ACB are equal; the triangle EBH is, therefore, similar to ACB, and if PF be drawn parallel to AE, and FG parallel to BH, the angle EFG will be equal to the angle ABC, and FG, EG will be given in magnitude; and, moreover, if GL be drawn parallel to CH it will be given by position, because the angle FGL is equal to the given angle BEA.

Again, because PF is parallel to AE, the angle PFB is equal to AEB (CHB), and the angle PFB is equal to CBA, therefore, adding the common angle APH, the triangles BFP, BHC, appear to be equiangular, and, consequently, $\text{BF} : \text{BH} :: \text{PF} : \text{CH}$; but the ratio of BF : BH is compounded of the ratios of HG : HE, and FG : EF; wherefore, $\text{PF} : \text{CH} :: \text{HG} \cdot \text{FG} : \text{HE} \cdot \text{EF}$. Take $\text{IG} : \text{PF} :: \text{EF} : \text{FG}$, then $\text{IG} : \text{CH} :: \text{HG} : \text{HE}$; and, by division, $\text{CD} : \text{IG} :: \text{HG} (\text{ID}) : \text{EG} (\text{ID being drawn parallel to EH, meeting CH in D})$; therefore, CD·DI is

is $\equiv \text{GI} \cdot \text{GE} \equiv$ to a given space, and the locus of the point C is a hyperbola, whose asymptotes are IL and ID.

Mr. Johnson likewise answered this Question.

XIV. QUESTION 44, by Mr. JOHN WRIGHT, *Norley.*

To find three square numbers in harmonical proportion such, that, if from each of them its respective root be taken, the remainders thence arising shall be rational square numbers.

SOLUTION, by the Proposer.

Let x^2 , y^2 , and z^2 be the three required numbers. Then by the property of harmonical proportion,

$$x^2 - y^2 : y^2 - z^2 :: x^2 : z^2; \text{ hence, } y^2 = \frac{2x^2 z^2}{x^2 + z^2},$$

$$\text{or } y = \frac{xz \sqrt{2}}{\sqrt{(x^2 + z^2)}} = \frac{(r^2 - s^2 + 2rs)(s^2 - r^2 + 2rs) l}{r^2 + s^2},$$

as will appear by substituting

$$\frac{(r^2 - s^2 + 2rs) l}{r^2 + s^2} \text{ for } x, \text{ and } \frac{(s^2 - r^2 + 2rs) l}{r^2 + s^2} \text{ for } y.$$

Assume now

$$l = \frac{(r^2 + s^2) n^2}{(2n - m) n}, \text{ and putting } a = r^2 - s^2 + 2rs;$$

$b = s^2 - r^2 + 2rs$, and $c = r^2 + s^2$; the expressions for x , y , and z above, will be transformed into

$$\frac{n^2}{(2n - m) m}, \frac{bn^2}{(2n - m) cm}, \text{ and } \frac{b^2 n^2}{(2n - m) am} \text{ respectively.}$$

Now, per quest. $x^2 - x$, $y^2 - y$, and $z^2 - z$ are all to be square numbers:

$$\text{but } x^2 - x = \frac{n^2 (n + m)^2}{m^2 (2n - m)^2} \text{ is already a square number,}$$

$$\text{and } y^2 - y = \frac{n^2}{(2n - m)^2 c^2 m^2} \times \left\{ b^2 n^2 - (2n - m) bcm \right\}$$

will be a square, when $b^2 n^2 - (2n - m) bcm$ is so: to effect which,

(N 2)

which, assume $b^2 n^2 \rightarrow (2n - m) \cdot bcm = (bn - pm)^2$; which will give $m = 2bn \times \frac{c-p}{bc-p^2}$. This value of m being written instead of it in the expression for z last found, we shall get

$$z = \frac{(bc - p^2)^2}{4ap(c-p)(b-p)}, \text{ and hence}$$

$$z^2 - z = \frac{(bc - p^2)^2}{16a^2 p^2 (c-p)^2 (b-p)^2} \times$$

$$\left\{ (bc - p^2)^2 - 4ap(b-p)(c-p) \right\}$$

which must be a square number, consequently $(bc - p^2)^2 - 4ap(b-p)(c-p) = b^2 c^2 - 4apbc - (2bc - 4ac - 4ab)p^2 - 4ap^3 + p^4 = \text{a square} = (bc + 2ap - p^2)^2$ suppose; whence is obtained

$$p = \frac{2bc}{c+b-a} = 2 \times \frac{s^4 - r^4 + 2rs(r^2 + s^2)}{3s^2 - r^2}$$

where r and s may be expounded by any two numbers taken at pleasure.

Ex. Gr. Take $r = 2$, and $s = 3$; then

$$m = \frac{1118264}{78455}, \text{ also } x = \frac{6155187025}{5041563384},$$

$$y = \frac{6155187025}{3855313176}, \text{ and } z = \frac{6155187025}{2075937864}.$$

XV. QUESTION 45, by X. Y.

It is required to investigate a rule for finding the area of any polygon whatever by means of its sides and angles,

SOLUTION, by Mr. LOWRY.

The investigation of a general rule for finding the area of any rectilineal polygon, by means of its sides and angles, will be much facilitated by the following Lemmas:

1st. Let

1st. Let a, b, c be any three arcs, then,

$$\sin(b+c) = \frac{\sin a \sin c + \sin b \sin(a+b+c)}{\sin(a+b)}.$$

It is well known that the $\sin(b+c) = \sin b \cos c + \cos b \sin c$; and, therefore, $\sin(a+b+c) = \sin a \cos(b+c) + \cos a \sin(b+c) = (\text{if } \cos b \cos c - \sin b \sin c \text{ be put for } \cos(b+c)) \sin a \cos b \cos c - \sin a \sin b \sin c + \cos a \sin(b+c)$; and, multiplying by $\sin b$ and adding $\sin a \sin c$, we have $\sin a \sin c + \sin b \sin(a+b+c) = \sin a \cos b \cos c \sin b + \sin a \sin c \cos^2 b + \cos a \sin b \sin(b+c) = \sin a \cos b (\sin b \cos c + \cos b \sin c = \sin(b+c)) + \cos a \sin b \sin(b+c) = \sin a \cos b + \cos a \sin b) (\sin(b+c)) = \sin(a+b) \sin(b+c)$; therefore, it is manifest that

$$\sin(b+c) = \frac{\sin a \sin c + \sin b \sin(a+b+c)}{\sin(a+b)}.$$

It might be shewn in a similar manner that

$$\sin(b-c) = \frac{\sin a \sin c - \sin b \sin(a-b+c)}{\sin(a-b)}.$$

2d. Let a, b, c, d be four arcs, then

$$\sin(b+c+d) = \frac{\sin a \sin(c+d) + \sin b \sin(a+b+c+d)}{\sin(a+b)}.$$

This is immediately deduceable from the 1st Lemma, by writing $(c+d)$ instead of c , and by that means may be extended to any number of arcs.

3d. If there are five arcs a, b, c, d, e , then

$$\sin(b+c+d+e) = \frac{\sin a \sin(c+d+e) + \sin b \sin(a+b+c+d+e)}{\sin(a+b)}.$$

And in general for any number of arcs $a, b, c, \&c. \dots n$,

$$\sin(b+c+\dots n) = \frac{\sin a \sin(c+d+\dots n) + \sin b \sin(a+b+c+\dots n)}{\sin(a+b)}.$$

To proceed with perspicuity we shall first find the area of a trapezium or polygon of four sides. Let two of the sides, as AB, DC (fig. 73, pl. 3.) be produced till they meet at b ; then it is evident that the trapezium $ABCD$ is equal to the difference of the triangles AbD, BbC . Now twice the area of the triangle AbD is $Ab \cdot bD \cdot \sin b = (AB + Bb) (DC + Cb) \sin b$, and twice the area of the triangle $BbC = Bb \cdot Cb \cdot \sin b$; and, therefore, their difference or twice the area of the trapezium is $= (AB \cdot DC + AB \cdot Cb + DC \cdot Bb) \sin b$; but $Cb \cdot \sin b$ is $= BC \sin B$, and $\sin BC \cdot \sin b = BC \cdot \sin C$, and $\sin b =$ the sine of

of the sum of the *exterior* angles $B + C$, therefore, twice the area of the trapezium is =

$$\begin{aligned} & AB \cdot BC \sin B \\ & + AB \cdot DC \sin (B + C) \\ & + BC \cdot DC \sin C. \end{aligned}$$

COR. Twice the area of the triangle ABC is =

$$\begin{aligned} & AB \cdot DC \cdot \sin (B + C) \\ & + BC \cdot DC \cdot \sin C. \end{aligned}$$

Again, the area of the pentagon ABCDE (fig. 74, pl. 3.) is equal to the sum of the areas of the trapezium ABCD, and the triangle DAE. Let the sides AB, DC be produced till they meet at b , then from the above we have twice the area of the trapezium ABCD =

$$\begin{aligned} & AB \cdot BC \cdot \sin B \\ & + AB \cdot DC \cdot \sin (B + C) \\ & + BC \cdot DC \cdot \sin C. \end{aligned}$$

And twice the area of the triangle DAE is =

$$\begin{aligned} & Ab \cdot DE \cdot \sin (b + D) \text{ or } \sin (B + C + D) \\ & + Db \cdot DE \cdot \sin D. \text{ That is =} \\ & AB \cdot DE \cdot \sin (B + C + D) \\ & + DC \cdot DE \cdot \sin D \\ & + Bb \cdot DE \cdot \sin (B + C + D) \\ & + Cb \cdot DE \cdot \sin D. \end{aligned}$$

But $Bb = \frac{BC \cdot \sin C}{\sin (B + C)}$, and $Cb = \frac{BC \cdot \sin B}{\sin (B + C)}$, there-

fore, the last two terms together become

$BC \cdot DE (\sin C \cdot \sin (B + C + D) + \sin B \sin D) = \sin (B + C)$
(by the 1st Lemma) $BC \cdot DE \cdot \sin (C + D)$. Now the various terms being collected, and arranged according to the order of the sides, we have twice the area of the pentagon =

$$\begin{aligned} & AB \cdot BC \cdot \sin B \\ & + AB \cdot DC \cdot \sin (B + C) \\ & + AB \cdot DE \cdot \sin (B + C + D) \\ & + BC \cdot DC \cdot \sin C \\ & + BC \cdot DE \cdot \sin (C + D) \\ & + DC \cdot DE \cdot \sin D. \end{aligned}$$

The same result will be obtained with equal facility by finding the difference between the trapezium $AbDE$ and the triangle BbC .

COR. Twice the area of the triangle DAE is =

$$\begin{aligned} & AB \cdot DE \cdot \sin (B + C + D) \\ & + BC \cdot DE \cdot \sin (C + D) \\ & + DC \cdot DE \cdot \sin D. \end{aligned}$$

In a similar manner we obtain the area of a hexagon by finding the sum of the areas of the pentagon ABCDE (fig. 75, pl. 3.) and the triangle AFE; for by the last rule twice the area of the pentagon is =

$$\begin{aligned} & AB \cdot BC \cdot \sin B \\ & + AB \cdot CD \cdot \sin (B + C) \\ & + AB \cdot DE \cdot \sin (B + C + D) \\ & + BC \cdot CD \cdot \sin C \\ & + BC \cdot DE \cdot \sin (C + D) \\ & + CD \cdot DE \cdot \sin D. \end{aligned}$$

And twice the area of the triangle AEF =

$$\begin{aligned} & Ab \cdot EF \cdot \sin (B + C + D + E) \\ & + Db \cdot EF \cdot \sin (D + E) \\ & + DE \cdot EF \cdot \sin E \\ & \text{equal } AB \cdot EF \cdot \sin (B + C + D + E) \\ & + DC \cdot EF \cdot \sin (D + E) \\ & + DE \cdot EF \cdot \sin E \\ & + Bb \cdot EF \cdot \sin (B + C + D + E) \\ & + Cb \cdot EF \cdot \sin (D + E). \end{aligned}$$

and writing for Bb and Cb their respective values $\frac{BC \cdot \sin C}{\sin (B + C)}$

and $\frac{BC \cdot \sin B}{\sin (B + C)}$ the sum of the last two expressions becomes

$$BC \cdot EF \cdot \sin C \cdot \sin \frac{(B + C + D + E) + \sin (D + E) \sin B}{\sin (B + C)} =$$

(by the 2d Lemma) to $BC \cdot EF \cdot \sin (C + D + E)$, and collecting and properly arranging the several terms we have twice the area of the hexagon =

$$\begin{aligned} & AB \cdot BC \cdot \sin B \\ & + AB \cdot CD \cdot \sin (B + C) \\ & + AB \cdot DE \cdot \sin (B + C + D) \\ & + AB \cdot EF \cdot \sin (B + C + D + E) \\ & + BC \cdot CD \cdot \sin C \\ & + BC \cdot DE \cdot \sin (C + D) \\ & + BC \cdot EF \cdot \sin (C + D + E) \\ & + CD \cdot DE \cdot \sin D \\ & + CD \cdot EF \cdot \sin (D + E) \\ & + DE \cdot EF \cdot \sin E. \end{aligned}$$

The same result may also be obtained by taking the difference of the areas of the pentagon $AbDEF$, and the triangle BbC ; and we have also this

COR. Twice the area of the triangle AEF is =

$$\begin{aligned} & AB \cdot EF \cdot \sin (B + C + D + E) \\ & + BC \cdot EF \cdot \sin (C + D + E) \\ & + CD \cdot EF \cdot \sin (D + E) \\ & + DE \cdot EF \cdot \sin E. \end{aligned}$$

In a like manner may the area of a heptagon be found, by finding the sum of the areas of the hexagon and the adjacent triangle, and from thence the area of an octagon, and so on for any other polygon.

The law of continuation is, however, sufficiently obvious from what has been done above, to enable us to extend the rule to any polygon whatever, without further investigation, and it may be expressed as follows :

Twice the area of any polygon is equal to the sum of the rectangles of its sides, except one, taken two and two, by the sines of the sums of the *exterior* angles contained by these sides.

In the application of the rule to any practical purpose, it will be most convenient to divide the polygon into two parts by a diagonal, and compute the area of each part separately.

Thus if the trapezium be divided into two triangles by the diagonal AC, twice the area will be equal to

$$AB \cdot BC \cdot \sin B \\ + CD \cdot AD \cdot \sin D.$$

And if the pentagon be divided into the trapezium ABCD, and the triangle ADE, we have twice the area =

$$AB \cdot BC \cdot \sin B \\ + AB \cdot DC \cdot \sin (B + C) \\ + BC \cdot DC \cdot \sin C \\ + DE \cdot AE \cdot \sin E.$$

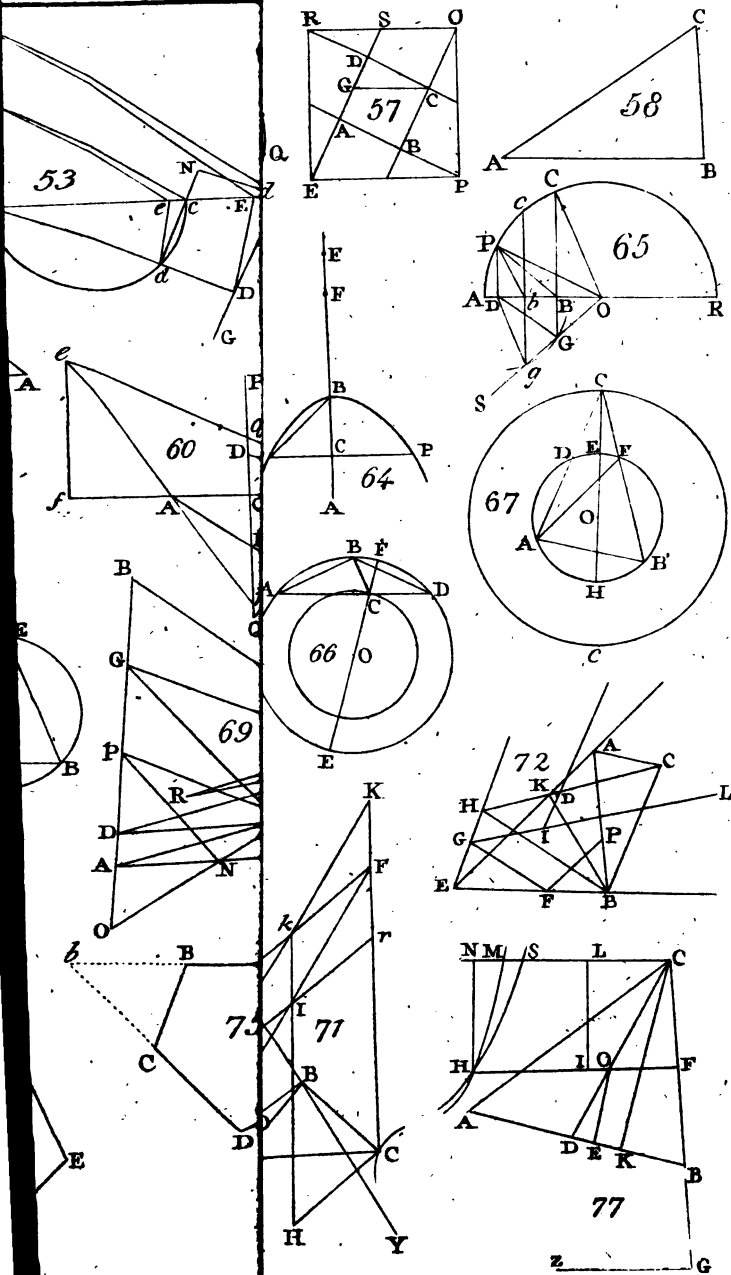
In like manner the hexagon may be divided into two trapeziums, and then twice the area will be =

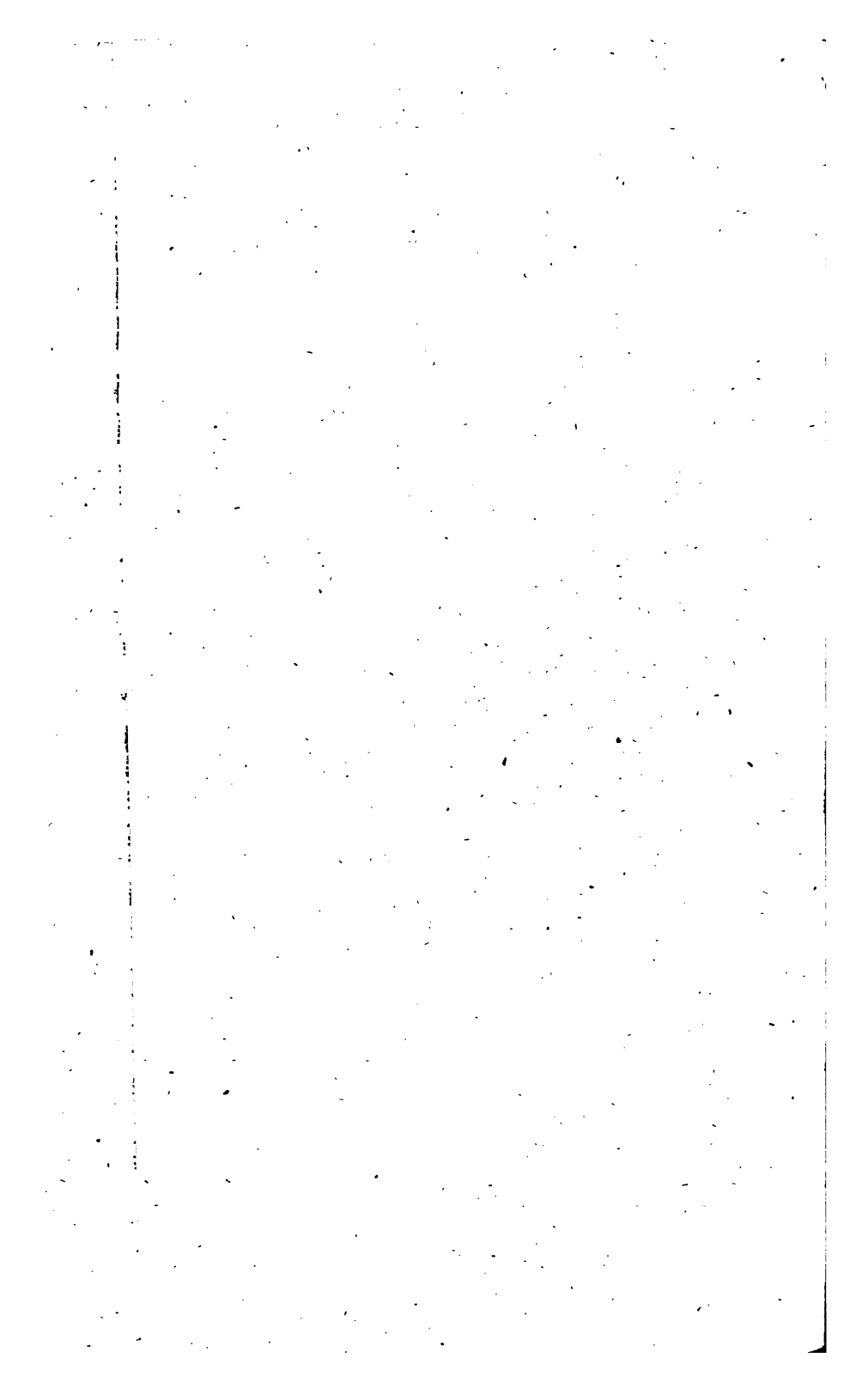
$$AB \cdot BC \cdot \sin B \\ + AB \cdot DC \cdot \sin (B + C) \\ + BC \cdot DC \cdot \sin C \\ + DE \cdot EF \cdot \sin E \\ + DE \cdot AF \cdot \sin (E + F) \\ + EF \cdot AF \cdot \sin F.$$

XVI. QUESTION 46, by NAUTICUS.

Suppose a great circle to pass through a given point of the equator and make with it a given angle, it is required to find its projection on Mercator's Chart.

SOLUTION,





SOLUTION, *by Mr. WALLACE, R. M. College.*

Let the curve, whose nature is sought, be referred to the straight line representing the equator, as an axis; and let the origin of the abscissas be at the given point in which the great circle meets the equator.

Let α denote the given angle; let ϕ denote an arch of the equator between the given point and a meridian passing through any variable point in the great circle, and ψ the latitude of that point: put x and y for the straight lines in the chart which represent the arches ϕ and ψ , or for the co-ordinates of a point in the curve whose latitude is ψ and longitude ϕ . Then, from the nature of the projection,

$$x = \phi, \quad y = \frac{\dot{\psi}}{\text{Cof. } \psi}.$$

And from spherical trigonometry,

$$\tan. \psi = \tan. \alpha. \sin. \phi;$$

and from these equations we are to deduce the relation between x and y .

Taking the fluxion of this last equation, and putting $\tan. \alpha = a$, we have

$$\frac{\dot{\psi}}{\text{Cof.}^2 \psi} = a \dot{\phi} \text{ cof. } \phi, \text{ and, therefore, } y = a \dot{\phi} \text{ cof. } \phi \text{ cof. } \psi.$$

But from the equation, $\tan. \psi = a \sin. \phi$, we find

$$\text{Cof. } \psi = \frac{1}{\sqrt{(1 + a^2 \sin.^2 \phi)}}.$$

$$\text{Therefore, } y = \frac{a \dot{\phi} \text{ cof. } \phi}{\sqrt{(1 + a^2 \sin.^2 \phi)}}.$$

and taking the fluents so that $y = 0$, when x or $\phi = 0$,

$$y = \text{hyp. log. } \left\{ a \sin. \phi + \sqrt{(1 + a^2 \sin.^2 \phi)} \right\}.$$

Hence the nature of the curve may be expressed thus:

Let ϕ be any arch of a circle to radius 1.

Take $x = \phi$,

$$y = \text{hyp. log. } \left\{ a \sin. \phi + \sqrt{(1 + a^2 \sin.^2 \phi)} \right\}.$$

Then x and y are co-ordinates to a point in the curve.

XVII. QUESTION 47, by GEOMETRICUS.

Let A, B be two given points in AB a straight line given by position, and let C, D be two given points without that line; let CV, DV be drawn, meeting AB in F and G, so that the rectangle $AF \times BG$ is given; It is required to determine the Locus of the point V.

SOLUTION, by Mr. LOWRY.

ANALYSIS. Join AC, BD, (fig. 78, pl. 4.) and draw DE parallel to AC meeting CI drawn parallel to AB in I. Let the rectangle $AC \cdot CR$ be taken equal to the given rectangle $AF \cdot BG$, draw SRQ parallel to AC, and take $RQ = EB$; draw CQ to meet EI produced in K, and DP parallel to AB meeting AC in P; also draw VL parallel to AC meeting PD in H, CI in O, and CK in L; then the triangles ACF, CRS, being similar, $AC \cdot CR = AF \cdot RS$, and by the hypothesis $AC \cdot CR = AF \cdot BG$, therefore RS is $= BG$, and RQ is $= BE$, therefore, SQ is $= EG$. Again, by similar triangles, $DE : EG = SQ :: VH : HD$, and $SQ : CR :: VL : CO = PH$; therefore, $DE : CR :: VH \cdot VL : DH \cdot HP$. Now in the trapezium CPDK, VL is drawn parallel to the sides CP, DK, and PD parallel to another of the sides AE, and the rectangles of the segments $VH \cdot VL$, $DH \cdot HP$ are in a given ratio, namely, that of DE to CR; therefore, the locus of the point V is a conic section, (see Hamilton's Conics, p. 56, B. i.) and is the celebrated *Locus Solidus* of the Ancient Geometricians, Newton in his Principia has given a solution to the general case of this Problem, and enumerated several of the particular ones, and the Problem has been still more distinctly considered by Dr. Pemberton, in an ingenious paper printed in the Philosophical Transactions for the year 1763. In what follows we shall endeavour to determine the species of the Locus in as brief and general a manner as possible, without entering into a particular detail of the various cases for each section; and in order to do this, let DP, KC, be produced till they meet at X, and XM be drawn to bisect CP, DK at Z and N, and meet the section in T and M; then it is evident that TM will be a diameter of the section, and VV an ordinate to that diameter, Now let MW be drawn parallel

to VY, meeting XD produced in W; then MW will be a tangent to the curve at M, and by the general property of the sections, the rectangles VH·VL, DH·HP, are as the squares of the femiaxes parallel to VL and PD, and (Hamilton, p. 31, B. i.) MW^2 is to $DW·WP$ in the same ratio; therefore, $MW^2 : DW·WP$ (VH·VL : DH·HP) :: DE : CR :: DE·CR : CR². Again, by similar triangles, $WX : 2MW :: CI : IK$; therefore, $WX^2 : MW^2 :: 4·CI^2 : IK^2 :: 4·CR^2 : RQ^2 = BE^2$, and (*ex aequo*) $WX^2 : DW·WP :: DE·CR : \frac{BE^2}{4}$.

Now it is evident from the figure of the sections, that in the ellipse WX^2 is greater than $DW·WP$, in the parabola equal, and in the hyperbola less; therefore the locus will be an ellipse parabola, or hyperbola, according as DE·CR is greater, equal to, or less than the square of half BE.

When the points BE coincide, the lines CI and CK will coincide, and the trapezium will become a parallelogram, and in this case the locus will be an ellipse, except when AC, DB, are perpendicular to AB, and the given rectangle = AC·DB, and then it becomes a circle whose diameter is CD, and from thence we derive the following properties of the circle.

1st. If from the extremities CD of the diameter of a circle, (fig. 79, pl. 4.) any two lines CV, DV, be inflected to the circumference, meeting the side AB of a rectangle described on CD in F and G: then the rectangle AF·BG will be equal to the square of AC or BD.

2dly. If from CD, the ends of the diameter, (fig. 80, pl. 4.) any two lines CV, DV, be inflected to the circumference, meeting any straight line in F and G, and the perpendiculars AC, DB be drawn to AB, then AF·BG will be = CA·DB.

There is also another case in which the locus is a circle, namely, when the diameter VM is perpendicular to the parallels AC, DK, and the given rectangle = to AC·DE. In this case (fig. 81, pl. 4.) PD is = CK, and if VY be produced to meet the circle again in a, VY will be = aY, VH = aL, and VL = aH; therefore, PH·HD = VH·Ha = VH·VL, as it ought to be. From the above we deduce the following curious theorem, viz.

If two equal chords PD, CK, be inscribed in a circle, and the lines CP, KD joining their extremities, be produced to meet any straight line AB drawn parallel to PD in A and E, and DB be drawn, making the angle EDB equal to the angle made by the chords produced till they meet; then if any two lines CV, DV, be inflected to the circumference meeting AE in F and G, the rectangle AF·BG will be equal to CA·DE.

There is also a case in which the locus becomes a straight line, viz. when the line joining the points CD, (fig. 82, pl. 4.) inter-

sects AB at E, making the rectangle AE·BE equal to the given one. In this case draw ICK parallel to AB, and let the line joining BD meet CKI in K; take AH = BE, and draw KHV, which will be the locus sought. For join AC, and produce VD to meet KC at I; then by reason of the parallels AH = BE : BG :: KC : KI :: HF : HG; therefore, by permutation and composition, AF : AH = BE :: HB = AE : BG; and therefore AF·BG is = AE·EB.

Messrs. Hopper and Smith also answered this Question.

XVIII. QUESTION 48, by Mr. W. SMITH, Liverpool.

Given the perimeter, the area, and the difference of the angles at the base of a plane triangle to construct it by the conic sections.

FIRST SOLUTION, by the Proposer.

CONSTRUCTION. Fig; 76, pl. 3. Form a rectangle HN MT such that its longest side HN may be equal to half the given perimeter, and its area equal to twice that of the given triangle; through the middle points of HT, NM draw CF, and through the point T with the asymptotes CF, CM describe the equilateral hyperbola TS. Divide TH in V so that TV may be to VH in the ratio of the line bisecting the vertical angle to the perpendicular, which ratio is given, because the difference of the angles at the base is given; now with the focus T, vertex V, and directrix HN, describe the hyperbola VS intersecting the former one in S, draw SQ parallel to CF meeting TF produced in Q, and join QC meeting TM in O, and on OQ describe a circle cutting FC in A, and CB drawn to make the angle QCB = ACQ in B; then if AB be drawn, ACB will be the triangle required.

DEMONSTRATION. By construction, the points A, O, B, Q are in a circle whose diameter is OQ; and the angle ACB is bisected by CQ, therefore, by a well known property of plane triangles, O is the centre of the inscribed circle, and Q the centre of a circle touching the base and the continuation of the two sides of the triangle ACB, and, therefore, the perimeter is equal to twice CF. Let SO produced meet CF in G, then SG is equal and parallel to FQ, and OG equal to the radius of the inscribed circle; and because the angles OSQ, OTQ are right ones, the points T, S are in the circumference of the circle described on OQ, and if TS be drawn, it will be a diameter, and therefore equal to OQ.

Now

Now by the property of the hyperbola and the construction OQ or TS : HQ :: TV : VH. Draw CD and QW perpendicular to AB, the latter meeting the circle in X, and join OX; let I be the point where AB meets CQ. Then it is well known that QX is = to QH; and, therefore, OQ : QX :: IC : CD :: TV : VH, that is, IC, the line bisecting the vertical angle, has to CD, the perpendicular, the given ratio; also the area of the triangle ACB is equal to the rectangle FC $\times$ OG, or FC $\times$ CM, that is, equal to the given space. *Q. E. D.*

SECOND SOLUTION, by Mr. LOWRY.

ANALYSIS. Let ABC (fig. 77, ph 3.) be the triangle required, DC the line bisecting the vertical angle, and O the centre of the inscribed circle; draw the radii OE, OF perpendicular to AB, BC, and on CB produced take CG = to half the given perimeter, and draw CK perpendicular to AB; then it is well known that FG = AB, and AB $\cdot$ CK = 2OF (or 2OE) $\cdot$ CG, therefore OF = OE is given in magnitude; moreover the angle DOE = DCK = half the difference of the angles at the base is given, and, therefore, OD is given in magnitude; and the ratio of DC to CK being the same as the ratio of DO to OE, it will be given; and, therefore, AB $\cdot$ DC will be = to 2OD $\cdot$ CG = to a given space = to MC $\cdot$ CG (if CM be drawn perpendicular to CG and = to 2OD). Now let FO be produced till FH = DC, and then BF $\cdot$ FH = AB $\cdot$ DC = CG $\cdot$ MC, therefore the locus of the point H is a hyperbola passing through M, and having CG and a line GZ drawn perpendicular to CG for its asymptotes. Again, bisect CM at L, and draw LI parallel to CB, meeting FH at I, and HN parallel to IL, meeting CM produced in N, and take LS = OF, then NH = CF and LN = IH = OC, and, therefore, LN² - LS² = HN² or 2LN $\cdot$ LS = HN², and consequently the locus of H is another equilateral hyperbola, whose vertex is S and centre L, and these two loci being described, their intersection will determine the point H, and then the sides of the triangle become known.

Messrs. Hopper and Johnson, also sent Solutions to this Question.

XIX. QUESTION 49, by ANALYTICUS.

Required the area of a curve whose equation is

$$y \sqrt{1 + ax^2 + x^4} = \frac{1 - x^2}{1 + x^2}.$$

SOLUTION,

SOLUTION, by the Proposer.

From the given equation we have

$$y = \frac{1-x^2}{1+x^2} \times \frac{1}{\sqrt{(1+ax^2+x^4)}} = \frac{1-x^2}{(1+x^2)x} \times \frac{1}{\sqrt{(\frac{1}{x^2}+a+x^2)}}$$

and, therefore, $\dot{y}$ or the fluxion of the area is

$$\frac{(1-x^2)\dot{x}}{(1+x^2)x} \times \frac{1}{\sqrt{(\frac{1}{x^2}+a+x^2)}}. \text{ Now to transform this}$$

fluxion to one more convenient for integration, let us assume

$$z = \frac{1}{x} + x, \text{ then } z^2 = \frac{1}{x^2} + 2 + x^2 \text{ and } a - 2 + z^2 = \frac{1}{x^2} + a + x^2.$$

$$\text{Also } \dot{z} = \dot{x} - \frac{\dot{x}}{x^2} = \frac{(x^2 - 1)\dot{x}}{x^2} \text{ and } \frac{\dot{z}}{z} = \frac{(x^2 - 1)\dot{x}}{(1+x^2)x};$$

therefore, by substitution, $\dot{y} = \pm \frac{\dot{z}}{z \sqrt{(a-2+z^2)}}$, it being positive or negative according as x is greater or less than 1.

The fluent of this expression may be found, as in Examp. 3, Art. 303, Simpson's Fluxions, and when a is greater than 2, it is =

$$\frac{1}{\sqrt{(a-2)}} \times \text{h. l.} \frac{\sqrt{(a-2+z^2)} - \sqrt{(a-2)}}{z} = \frac{1}{\sqrt{(a-2)}} \times \text{h. l.}$$

$$\frac{\sqrt{(\frac{1}{x^2} + a + x^2)} - \sqrt{(a-2)}}{\frac{1}{x} + x}. \text{ But when } a \text{ is less than 2, the}$$

expression $\frac{1}{\sqrt{(a-2)}}$ will be impossible; and in this case the fluent will be expounded by

$$\frac{1}{\sqrt{(a-2)}} \times \text{circ. arc, whose secant} = \frac{z}{\sqrt{(a-2)}} \left(= \frac{1+x^2}{x\sqrt{(a-2)}} \right) \text{ to rad. unity.}$$

XX. QUESTION 50, by AMICUS.

If a represent any arch of a circle, that arch is equal to the infinite product

$$\sin. a \times \sec. \frac{1}{2} a \times \sec. \frac{1}{4} a \times \sec. \frac{1}{8} a \times \sec. \frac{1}{16} a \times \&c.$$

As this very curious expression for a circular arch does not seem generally known, it is required to find its demonstration.

SOLUTION,

SOLUTION, by Mr. JOHN DAWES, *Birmingham.*

It is well known that $\sin. 2a = 2 \sin. a \cos. a$, and if for a we write the successive values $\frac{1}{2}a, \frac{1}{4}a, \frac{1}{8}a, \dots, \frac{1}{2^n}a$, we obtain

$$1^{st} \quad \sin. a = 2 \sin. \frac{1}{2}a \cos. \frac{1}{2}a$$

$$2^{d} \quad \sin. \frac{1}{2}a = 2 \sin. \frac{1}{4}a \cos. \frac{1}{4}a$$

$$3^{d} \quad \sin. \frac{1}{4}a = 2 \sin. \frac{1}{8}a \cos. \frac{1}{8}a$$

$$4^{th} \quad \sin. \frac{1}{8}a = 2 \sin. \frac{1}{16}a \cos. \frac{1}{16}a$$

$$\text{And} \quad \sin. \frac{1}{2^n}a = \sin. \frac{1}{2^n}a \cos. \frac{1}{2^n}a.$$

Now if the sine $\frac{1}{2}a$ as found in the second equation, be substituted for $\sin. \frac{1}{2}a$ in the first equation, we have $\sin. a = 4 \sin. \frac{1}{4}a \cos. \frac{1}{4}a \cos. \frac{1}{4}a$ and substituting for $\sin. \frac{1}{4}a$, its value found in the third equation, we have $\sin. a = 8 \sin. \frac{1}{8}a \cos. \frac{1}{8}a \cos. \frac{1}{8}a \cos. \frac{1}{8}a$, and by continuing this substitution we at last obtain

$$\sin. a = 2^n \sin. \frac{1}{2^n}a \cos. \frac{1}{2^n}a \cos. \frac{1}{2^n}a \cos. \frac{1}{2^n}a \dots \cos. \frac{1}{2^n}a.$$

Now when n is indefinitely great, the arc $\frac{1}{2^n}a$ will be indefinitely small, and therefore may be considered as equal to its sine, and the $\cos. \frac{1}{2^n}a$ will be equal to the radius (1); therefore, the above expression becomes

$$\sin. a = a \cos. \frac{1}{2}a \cos. \frac{1}{4}a \cos. \frac{1}{8}a \cos. \frac{1}{16}a \&c.$$

$$\text{or } a = \frac{\sin a}{\cos. \frac{1}{2}a \cos. \frac{1}{4}a \cos. \frac{1}{8}a \cos. \frac{1}{16}a \&c.} =$$

$$\sin. a \times \frac{1}{\cos. \frac{1}{2}a} \times \frac{1}{\cos. \frac{1}{4}a} \times \frac{1}{\cos. \frac{1}{8}a} \times \frac{1}{\cos. \frac{1}{16}a} : \text{but in any arch}$$

$$\text{the secant is } = \frac{\text{rad.}^2}{\cos.} = \frac{1}{\cos.} \text{ therefore by substitution.}$$

$$a = \sin. a \times \sec. \frac{1}{2}a \times \sec. \frac{1}{4}a \times \sec. \frac{1}{8}a \times \sec. \frac{1}{16}a \&c.$$

The investigation of this expression for a circular arc is given by Lacroix, in the Introduction to his *Traité du Calcul. différentiel et intégral*, page 79; but the invention belongs to Euler.

XXI. QUES:

XXI. QUESTION 51, by CURIOSUS.

In a work entitled "*Hemisphaerium Dissectum*:" Authore *Ricardo Albio*, 4to. *Rome*, 1648, it is said that the convex surface of a scalene cone is equal to a circle whose radius is a mean proportional between $\frac{1}{2}\sqrt{(h^2 + r^2)} + \frac{1}{2}S + \frac{1}{2}s$ and r , where h is the cone's perpendicular height, r the radius of the base, S and s the greater and less slant heights. How is this rule derived?

SOLUTION, by Mr. LOWRY.

Let AIEZ (fig. 83, pl. 4.) represent the circular base of the cone, C its vertex, and AC, CE the greater and less slant sides respectively; draw the diameter AOEG and demit the perpendicular CG, which will evidently represent the perpendicular altitude of the cone. Let Ii be two points in the circumference of the base, indefinitely near to each other, join CI, Ci, and draw the tangent IS meeting AG (produced if necessary) in S, and let CK be drawn at right angles to IS.

Then because the arc Ii is indefinitely small, the triangle ICi will be equal to the arc Ii, multiplied by half the perpendicular CK, or taking (on the diameter produced) AH, Ah equal to the arcs EI, Ei respectively, and the perpendicular HK = half CK, the triangle ICi, will be equal to the rectangle hHK; and as the surface of the cone is made up of an indefinite number of these little triangles ICi, or their equals the rectangles hHK, it follows that if AF be taken equal to the semi-circumference AIE, and ordinates be taken at every point H corresponding to every point I round the circumference, equal to half the perpendicular CK drawn to the tangent at that point, the area of the curve AIE will be equal to half the surface of the cone. To determine the exact area of this curve, is however no easy task, nor would it be of any use in the investigation of the rule mentioned in the question, which is only an approximation, and seems to have been derived from some such method as follows:

When the points AI coincide, it is evident that CA and CK will also coincide, and therefore FT the ordinate corresponding to the point F (or A) will be equal to half CA = $\frac{1}{2}S$, and in the same way it appears that AL the ordinate corresponding to the point A (or E) is equal to half CE = $\frac{1}{2}s$. Let AF be bisected at H' and draw the ordinate H' K', then H' is the point corresponding

to I the middle of the arc AIE, and in this case the tangent IS will be parallel to the diameter AE, and the line joining GK will be equal and parallel to IO the radius of the cone's base; therefore CK will be $= \sqrt{(GC^2 - GK^2)}$ and $H'K' = \frac{1}{2} \sqrt{(CG^2 - GK^2)} = \frac{1}{2} \sqrt{h^2 + r^2}$. Now it is very evident that the area of the curve will be nearly equal to the sum of the two trapezoids TFH'K' and H'K'EL, the small defect in the one being nearly compensated by the excess in the other, therefore the area may be considered as

equal to $\left(\frac{FT}{2} + H'K' + \frac{AL}{2} \right) \frac{FA}{2} = \left(\frac{1}{2} \sqrt{h^2 + r^2} + \frac{S}{4} + \frac{s}{4} \right) \times \frac{FA}{2}$, and therefore the whole convex surface of the cone will

be equal to twice that space $= \left(\frac{1}{2} \sqrt{h^2 + r^2} + \frac{S}{4} + \frac{s}{4} \right) \times$

$r \times 3.14159$ &c. (FA being equal to $r \times 3.14159$) which is evidently equal to a circle whose radius is a mean proportional

between $\frac{1}{2} \sqrt{h^2 + r^2} + \frac{S}{4} + \frac{s}{4}$ and r .

This Rule will be sufficiently exact for practical purposes, especially when the slant sides are nearly equal; and in cases where greater accuracy is necessary, the area may be approximated to any degree of exactness, and with the greatest facility, by the method of equidistant ordinates.

XXII. QUESTION 52, by ALIQUIS.

Having given the base of a triangular pyramid, and the three plane angles at the vertex; to find the other dimensions?

SOLUTION, by Mr. LOWRY.

To render the Solution of this Problem as simple as possible, let us conceive the surface of the pyramid to be expanded on a plane as represented in fig. 84, pl. 4; then we shall have given the angles BAC, CAD, DAE, and the opposite sides BC, CD, DE, to find the other sides AB, AC, AD and AE, whereof AB, and AE are equal. Now assume AF = AI of any given length, and draw FG, GH, HI parallel to BC, CD and DE respectively, and drop the perpendiculars FL, IK; then the figure FGHIA will be similar to BCDEA, and therefore the lines FG, GH, HI will have the same ratio as the given lines BC,

CD, DE; and because AF = AI is given, the lines FL, AL, IK and AK will also be given. Now put FL = a , AL = b , IK = c , AK = d , $\cos. \angle EDC = v$, LG = x , KH = y and let m , n , r represent the ratios of the squares of FG, GH and HI respectively; then $FG^2 = a^2 + x^2$ and $IH^2 = c^2 + y^2$ therefore, $a^2 + x^2 : c^2 + y^2 :: m : r$

$$or\ x^2 = \frac{m}{r} y^2 + \frac{m}{r} c^2 - a^2.$$

Again, by trigonometry, $GH^2 = GA^2 + HA^2 - 2v.GA.AH$, and if for AG, and AH, there be written $b + x$, and $d + y$, and w^2 be put for $b^2 + d^2 - 2vbd$, we have $GH^2 = x^2 + y^2 + 2(b - vd - vy)x + 2(d - vb)y + w^2$ and, therefore, $x^2 + y^2 + 2(b - vd - vy)x + 2(d - vb)y + w^2 : c^2 + y^2 :: n : r$, and this by reducing becomes $2(b - vd - vy)x = \frac{n}{r} c^2 + \frac{n - r}{r} y^2 - x^2 - 2(d - vb)y - w^2$, and substituting for x its value as found above, we have,

$$2(b - vd - vy) \cdot \left(\frac{m}{r} y^2 + \frac{m}{r} c^2 - a^2 \right)^{\frac{1}{2}} = \frac{n - m}{r} c^2 + \left(\frac{n - r - m}{r} \right) y^2 - 2(d - vb)y - w^2 + a^2$$

which being expanded and properly reduced, we may obtain the value of y by a biquadrate, when the given quantities are expressed in numbers, and from thence the other parts will readily become known.

It is clear from the above, that the problem cannot in general be constructed by plane Geometry, but it may be easily done by the conic sections; the locus expressed by each of the above equations being an hyperbola.

Or if it should be more desirable to have the resulting equation expressed in terms of the quantities given in the question, without first finding the dimensions of a similar pyramid, we may proceed as follows: Put x , y , and z , for the three slant sides, m , n and s for the cosines of the given angles, and a , b , c , for the sides of the base opposite to those angles; then by trigonometry we have the following equations,

$$x^2 + y^2 - 2xym = a^2$$

$$y^2 + z^2 - 2yzn = b^2$$

$$x^2 + z^2 - 2xzs = c^2.$$

Now let us assume $y = wx$ and $z = vx$, then our equations become

$$x^2 (1 + w^2 - 2wm) = a^2$$

$$x^2 (w^2 + v^2 - 2wvn) = b^2$$

$$x^2 (1 + v^2 - 2vs) = c^2.$$

And from hence we have three values of x^2 which being equated we obtain

$$a^2 (w^2 + v^2 - 2wvn) = b^2 (1 + w^2 - 2wm)$$

$$a^2 (1 + v^2 - 2vs) = c^2 (1 + w^2 - 2wm)$$

or,

or,

$$\begin{aligned} a^2 v^4 - 2wna^2 v - b^2(1 + w^2 - 2wm) - a^2 w^2 &= 0 \\ a^2 v^2 - 2sa^2 v - c^2(1 + w^2 - 2wm) - a^2 &= 0. \end{aligned}$$

Now to exterminate one of the unknown quantities, as v , put $P = 2wna^2$, $Q = b^2(1 + w^2 - 2wm) - a^2 w^2$, $R = 2sa^2$ and $S = c^2(1 + w^2 - 2wm) - a^2$; then, we have

$$\begin{aligned} a^2 v^2 - Pv - Q &= 0 \\ a^2 v^2 - Rv - S &= 0. \end{aligned}$$

Subtract the second equation from the first, and there results

$$(R - P)v + S - Q = 0; \text{ or, } v = \frac{Q - S}{R - P}.$$

Again subtract the second equation multiplied by Q , from the first equation multiplied by S , and divide the remainder by v , and we have

$$a^2(Q - S)v - PS + QR = 0; \text{ or, } v = \frac{PS - QR}{a^2(Q - S)},$$

and equating these values of v , we have

$a^2(Q - S)^2 = (PS - QR) \cdot (R - P)$; which, by substituting for P , Q , R and S , their respective values become

$$[(b^2 - c^2)(1 + w^2 - 2wm) + 1 - w^2]^2 = [(wnb^2 - ac^2)(1 + w^2 - 2wm) + (sp^2 - con)a^2]^2 \cdot 4(s - con)a^2.$$

Where the value of w may be found by an equation of the 4th power, and from thence n and x (and of course y and z) become known.

XXIII. QUESTION 53, by Mr. WM. WALLACE, of the Royal Military College.

Let AB , AC be two straight lines given by position, and let DE be a line of a given length terminated by them at D and E , let DV , EV , be perpendiculars to the lines AB , AC , meeting in V . It is required to determine the *Locus* of the point V .

FIRST SOLUTION, by Miss MARIA PORTER, London.

Suppose that AB , AC , (fig. 85, pl. 4.) are the two lines placed in their given position, DE the given line drawn as per question, and DV , EV the two perpendiculars. Then, since the $\angle$ s ADV , AEV are right $\angle$ s, the $\angle A + \angle DVE$ must be equal to two right angles; but the $\angle A$ is given, therefore, the $\angle DVE$ is also given, it being equal to the supplement of the $\angle A$.

(P 2)

Take

Take PQ = the given line DE and (Euc. 33. b. iii.) describe a segment of a circle PmQ to contain an angle equal to the given angle DVE ; draw Pm to make the angle $QPm = DEV$; and join Qm . Let o be the centre of the circle, draw the diameter mon and join Pn , Qn ; and draw AV : then, because the angle $DEV = mPQ$, and $\angle EVD = PmQ$, therefore, the triangles PmQ , and EVD are equiangular; therefore, because $PQ = DE$ the two triangles are equal in every respect, and $Pm = EV$ and $mQ = VD$. And because the two right angles mPn and VEA are equal and the two parts mPQ and VED likewise equal, the remainders AED and nPQ are equal: Hence it is manifest that the two triangles AED and nPQ are equiangular, therefore, they are equal in all respects, because $ED = PQ$. And because $EV = Pm$ and $EA = Pn$ and the angles at E and P are right angles, therefore, AV is equal to mn , that is to the diameter of a given circle: therefore, the locus of the point V is a given circle of which A is the centre.

SECOND SOLUTION, by Mr. THOS. MYERS, *Hammer Smith.*

Fig. 85, pl. 4. The angles at D and E being right angles, the points A , D , V , E , are in the circumference of a circle whose diameter is AV , and if the diameter DF be drawn meeting the circle again in F , and EF be joined, the angle DFE is equal to the given angle DAC , and DEF is a right angle, and DE is given in magnitude; therefore, DF and its equal AV will be given, and, consequently, the locus of the point V will be a circle whose centre is A and radius AV .

THIRD SOLUTION, by PORUS.

Draw EG (fig. 86, pl. 4.) perpendicular to DE and VG perpendicular to VD meeting EG in G ; join DG , AV , and taking $AC = AV$ let fall CB perpendicular to AB . Then will V , G , E , D , be points in a circle whose diameter is DG , and A , D , V , E points in a circle whose diameter is AV . Consequently, the angles VEG , VDG , and AVD , AED are equal, but VEG is equal to AED , because GED , VEA are equal by construction; therefore, in the triangles DVG , DAV , the angles VDG , AVD and GVD , ADV are equal, and the side DV is common to both; hence DG is equal to AV . Now the angle EVD or its equal BAC is equal to EGD , the angle ABC equal to DEG , and AC equal to AV or DG by the construction; consequently, BC is equal to DE , but DE and the angles BAC , ABC are constant
by

by the hypothesis and construction, wherefore, AC or its equal AV is constant, and the locus of V is a circle.

COR. It is evident that if DV, VE had been drawn so as to make the angles ADV, CEV equal, the locus of the point of concurrence V would have also been a circle, for if the angle DEG be made equal to AEV, the angle DVG equal to CEG, and ABC equal AEV, the steps of the demonstration will be exactly the same.

Solutions were also received from Messrs. Bazley, Dawes, Hopper, Johnson, Lowry, and Smith.

XXIV. QUESTION 54, *by* URSA MINOR.

Three straight lines being given by position, and three points being also given, it is required to describe a rectilinear figure, whose sides may pass through the given points, and whose angles may be upon the lines given by position.

SOLUTION, by Mr. LOWRY, R. M. College.

The Analysis which I have given to the 42d Question, will assist us in resolving this problem, and also in describing a rectilinear figure of any number of sides whatever, so that its angular points may fall on straight lines given by position, and its sides pass through given points.

Let AB, BC, AC, (fig. 87, pl. 4.) be the three straight lines given by position; D, E, F, the given points, and IGK the triangle to be described, so as to have its angular points on the straight lines given by position, and its sides passing through the given points. Now proceeding exactly as in that Analysis, (supposing the points H, C to coincide,) we may find a point R, and the position of the straight line RS, such, that drawing KG, IG, through the points E, F to intersect in AB, and meet RS in S, and AC in I, KS may be equal to IC. And in the same manner we may find a point L, and the position of the straight line LM, such, that drawing GK, ID through the points E, D to intersect in BC, and meet RS in S, and LM in M, RS may be equal to LM; then it is obvious that the lines RS, LM, and IC will be equal, when the triangle is described in the manner required, and, therefore, we have only to draw IK through the point D, to meet AC in I, and LM in M, so that IC may be equal to LM, which is a particular case of the *Sectio Rationis* of Apollonius.

Again,

Again, let it be required to describe a rectilinear figure *abcde*, (fig. 88, pl. 4.) having its angles on the straight lines AE, AB, BC, CD, DE given by position, and its sides passing through the given points F, G, H, I, K.

Find as before, the point L, and the position of the straight line LS, such, that drawing *ab*, *bc*, through the points F, G to intersect in AB, and meet AE in *a*, and LS in S, *aE* may be equal to LS. Also find the point M, and the position of the straight line MT, such, that drawing *bc*, *cd*, through the points G, H to intersect in BC, and meet LS in S, and MT in T, LS may be equal to MT. Likewise find the point N, and the position of the straight line NU, such, that drawing *cd*, *de* through the points H, I to intersect in DC, and meet MT in T, and NU in U, MT may be equal to NU. And also find the point O, and the position of the straight line OV, such, that drawing *de*, *ae*, through the points I, K to intersect in DE, and meet NU in U, and OV in V, NU may be equal to OV. And in this manner we may proceed for any number of sides, by finding the position of as many subsidiary lines LS, MT, NU, VO, &c. less one, as the figure has sides; and then it is obvious that the lines *aE*, LS, MT, NU, OV will be equal, when the figure is described in the manner required, and, therefore, we have only to draw *ae*, to pass through K, and meet AE in *a*; and OV in V, so that *aE* may be equal to OV, and then the position of the figure will be determined.

It may not be improper to remark that the proposition which we derive from the Analysis to the 42d Question, viz: that RS is always equal to HI, may be enunciated in the porismatic form, and is then a particular case of the first porism of Euclid, as preserved by Pappus in his Mathematical Collections*, and which Dr. Simson has with great ingenuity, applied to the Solution of the Problem, from which it is supposed the porism was originally derived. The application we have made of it in the above Solution may serve as an additional specimen of the great use of porisms in the Analysis of difficult geometrical Problems.

Dr. Simson, at page 388 of his *Opera Reliqua*, has given a Solution to that particular case of the Problem when the given points are in the same straight line, and we shall now show how the principles he has there made use of, may be applied to the Solution of the general case when the points are in any other given position; and first it will be necessary to demonstrate the following proposition, which is nearly the same as Prop. 130, Lib. 7, of Pappus, and is that on which Dr. Simson founds his Analysis.

* See also Simson's Treatise de Porismatibus, Prop. 23, and Playfair's Paper on Porisms, reprinted in the "Mathematical Tracts and Selections."

PROP.

Let there be six points (fig. 89, 90, pl. 4.) A, B, C, D, E, F in the straight line AF, such, that $AD \cdot BC : AB \cdot DE :: CF : EF$, and let three straight lines YG, ZG, HG pass through the points D, E, F, and intersect in G; and from the points B, C any two straight lines BH, CH, be inflected to GH meeting YG in L, and ZG in K; then the points A, L, K will be in a straight line.

Through K draw PM parallel to AF meeting YG in M, BH in N, and HG in P; and draw EX parallel to CK meeting GH in X; then $CF : FE :: CH : EX :: CH \cdot KG : EX \cdot KG = KH \cdot GE$, but $KG : GE :: KM : ED$ and $CH : HK :: CB : KN$, therefore,

$CF : FE :: KM \cdot CB : ED \cdot KN$; but by hypothesis,

$CF : FE :: AD \cdot BC : AB \cdot DE$, therefore,

$AD : AB :: MK : NK$, and by division, &c.

$AB : NK :: BD : MN :: BL : LN$; and the angles

ABL, LNK are equal, therefore, the triangles ABL, LNK are equiangular, and, consequently, the points A, L, K are in a straight line.

This being premised, let A, B, V be the given points, (fig. 89.) GY, GZ, H the straight lines given by position, and LKH the required triangle, having its angular points on the given straight lines, and its sides passing through the given points. Join GH, and through the given points A, B, draw AF meeting GY in D, GZ in E, GH in F, and HKV produced in C; then it is manifest from the Prop. that $CF : FE :: AD \cdot BC : DE \cdot AB$; and if $AD : AB :: DE : EI$, we have $CF : FE :: BC : EI$, and by division $CE : EF :: BC - EI : EI$, and by permutation and composition, $BI : BC - EI :: IF : EI$, or, taking $BS = EI$, we have $BI \cdot EI = SC \cdot IF$, = to a given space; for, because AD, AB and DE are given, EI (and of course BI) will be given and also the points S, I; the problem is therefore reduced to this, viz. to inflect two straight lines VH & GH from the given points V, G, to the line H to intersect AF in C and F, so that the rectangle SC·IF may be equal to the given space BI·IE, which is the particular case specified at the end of my Solution to the 12th Question in the present number.

The lines GH, CH being thus drawn, and BH joined, it will appear by reversing the above Analysis, that $CF : FE :: AD \cdot BC : DE \cdot AB$, and, therefore, by the Prop. the points A, L, K are in a straight line, and, consequently, the triangle LKH is described in the manner required.

Mr. Smith also favoured us with a Solution to this Question.

XXV. QUESTION 55, by QUINBUS FLESTRIN,

Given the base and the length of a line from the vertical angle to a given point therein, to construct the triangle when the vertical angle is a maximum.

FIRST SOLUTION, by Mr. THOMAS HOPPER.

Let CB (fig. 91, pl. 4.) be the given base, and P the given point; with the centre P and distance equal to the given line describe a semicircle AFD meeting the base produced in A and D; take DE, such, that $DE : CD :: BD : AB - CD$, and from E draw EF to touch the semicircle at F; join BF, CF and BFC will be the triangle required.

Through the points BFC describe a circle; then because $DE : CD :: BD : AB - CD$, we have by composition $DE : CE :: BD : AC$, and again, by alternation and composition, $DE : BC :: CE : AE$; therefore, $BC \cdot CE = DE \cdot AE = EF^2$, and, consequently, EF is a tangent to both the circles, (Eu. 37. b.iii.) and, therefore, they touch each other at F, and the angle BFC will in that case be a maximum, as is well known,

SECOND SOLUTION, by Mr. COLLINS, *Kenfington*.

Let AB (fig. 92, pl. 4.) be the given base, D the given point therein, with which as a centre and radius $=$ the given line, drawn from the vertex to the given point, describe the circle aCb , then by Prop. 45, Simpson's Geometry, describe a circle to pass through A and B, and touch the other circle in C, join AC and BC, and ABC will be the triangle required.

DEMONSTRATION. By the construction the base AB, the segments AD, DB, and the line DC are all equal to the given ones. We have therefore only to prove that the angle ACB is a maximum. To do this, let any other lines be drawn to the circle as at c , and the angle ACB will be equal to the angle AcB , and, consequently, greater than the angle AdB formed by any other two lines Ad , Bd , drawn to any other point in the circle aCb .

THIRD

THIRD SOLUTION, *by Mr. THOMAS BAZLEY.*

ANALYSIS. Suppose it done, ABC, (fig. 93, pl. 4.) the required Δ , S the given point in the base, SC the given right line, and ACBK the circumscribing circle whose centre is O. Then, because the $\angle C$ is a maximum, the circle described from centre S with radius SC must touch the circumscribing circle in C. For if C be supposed to fall in any other point of the exterior circle than the point of contact, it is plain then, that the $\angle C$ would be less than the $\angle$ in the circumscribing circle, that is less than that at the point of contact. Now it is known that C, O, S are in the same right line: produce it to meet the circumscribing circle in R and drop OD perpendicular to AB. Then $AD = \frac{1}{2}AB$ is given, and since the point S is given, therefore SD is given; consequently, $AD^2 - DS^2 = AO^2 - OS^2 = CS \times SR$ is given. But CS is given, therefore SR is given, and, consequently, $\frac{CS - SR}{2} = OS$ is given, and the construction is very evident.

LIMITATION. It is requisite that SC be *greater* than either of the segments SA, SB, when the vertical $\angle$ is required to be a maximum; and *less* when a minimum.

Ingenious Solutions were likewise received from Messrs. Johnson, Lowry, Merones Minor, Myers, and Pickering.

XXVI. QUESTION 56, *by Mr. JAMES CUNLIFFE.*

Divide the sum of three square numbers in arithmetical proportion into three other square numbers in arithmetical proportion, and give an example in whole numbers as small as may be.

SOLUTION, *by Mr. JAMES IVORY, R. M. College.*

When three numbers are in arithmetical progression, it is easy to prove that the sum of all the three is triple of the middle number: therefore, if two sets of three numbers in arithmetical progression have the same sum, it is obvious that the middle term of each set must be the same; and, on the contrary, if the middle terms be the same, the sums of the two sets will likewise be the same.

Let then a^2, b^2, c^2 and a'^2, b'^2, c'^2 be two sets of square numbers in arithmetical progression, having the same middle term b^2 , and, consequently, the same sum $2b^2$: then

$$2b^2 = a^2 + c^2 = a'^2 + c'^2$$

or

$$b^2 = \left(\frac{a+c}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2 = \left(\frac{a'+c'}{2}\right)^2 + \left(\frac{a'-c'}{2}\right)^2.$$

It is manifest, therefore, that the Question is reduced to the finding of a square number that may be resolved into the sum of two squares twice. Attending to the principles laid down in the Solution of Question 23d, No. II. of the Repository, it will appear that the number sought must contain prime divisors of the form $4x+1$; and, as a least value is wanted, we may neglect all the other divisors which would only increase the number sought without rendering it more adapted to the Solution of the Question. We may infer likewise from the same principles, that no single prime number of the form $4x+1$ will satisfy the required conditions; because a prime number of the form $4x+1$, as well as its square, can be the sum of two squares only once. Taking then a number composed of two such prime factors, we shall readily find that the number itself is the sum of two squares twice, and that its square is the sum of two squares no less than four times. Thus if $p = a^2 + \beta^2$ and $p' = a'^2 + \beta'^2$: then

$$pp' = \begin{cases} (a\alpha' + \beta\beta')^2 + (a\beta' - \alpha'\beta)^2 = A^2 + B^2 \\ (a\alpha' - \beta\beta')^2 + (a\beta' + \alpha'\beta)^2 = C^2 + D^2. \end{cases}$$

And

$$p^2 p'^2 = \begin{cases} (A^2 - B^2)^2 + 4A^2 B^2 \\ (C^2 - D^2)^2 + 4C^2 D^2 \\ A^2 p'^2 + B^2 p'^2 \\ A'^2 p^2 + B'^2 p^2 \end{cases}$$

Hence it is evident that the number sought must be the product of two primes of the form $4x+1$; and that each such number will furnish no less than 6 Solutions of the Question, viz. as many Solutions as there are combinations of two things in four.

The two least primes of the form $4x+1$ are 5 and 13; their product is 65: now

$$\begin{aligned} 65^2 &= 33^2 + 56^2 \\ 65^2 &= 63^2 + 16^2 \\ 65^2 &= 52^2 + 39^2 \\ 65^2 &= 25^2 + 60^2. \end{aligned}$$

From these we derive the following sets of three-squares in arithmetical progression,

$$\begin{array}{rcl}
 23^2, & 65^2, & 89^2 \\
 47^2, & 65^2, & 79^2 \\
 13^2, & 65^2, & 91^2 \\
 35^2, & 65^2, & 85^2.
 \end{array}$$

The sum of each set being the same and equal to 3×65^2 ; and these sets of numbers furnish 6 different Solutions of the Question, by taking all their combinations, two and two.

But if we suppose $p = p'$, then the four forms for $p^2 p'^2$ will give only two forms for p^4 (vide Quest. 23, N. 2.): and thus the square of every prime of the form $4x + 1$ will give an example, admitting only one Solution of the Question. The least prime of the form $4x + 1$ is 5: its square is 25: and we have

$$\begin{array}{l}
 25^2 = 7^2 + 24^2 \\
 25^2 = 15^2 + 20^2
 \end{array}$$

whence we have

$$\begin{array}{rcl}
 31^2, & 25^2, & 17^2 \\
 35^2, & 25^2, & 5^2
 \end{array}$$

which are the least numbers that satisfy the Question.

XXVII. QUESTION 57, by Mr. CUNLIFFE.

Let a slender heavy inflexible rod AB, loaded with a weight, be placed with its upper end B leaning against an inclined plane DE, and its lower end A upon the curve DA, situated in a vertical plane at right angles to the inclined plane. It is required to investigate the nature of the curve DA, so that the rod may rest in equilibrio when placed in any inclination.

SOLUTION, by Mr. LOWRY, R. M. College.

Let dD (fig. 94, pl. 4.) be the given plane making any given angle with the horizontal plane IDK, DAQ a portion of the required curve, AB the given rod with the weight attached resting in equilibrio in any assumed position, and E their centre of gravity, then because AB is of a given length the parts AE, BE, will also be given.

Now by a well known principle in mechanics, when the rod rests in equilibrio, the centre of gravity will be at the lowest place, or the nearest to the horizontal plane IK that it can possibly get; and as the rod is to rest in equilibrio in whatever position it is placed, it is manifest that the centre of gravity must in every position be equally distant from the plane IK, for if it was not the rod would not be at rest, but would slide along the curve till it came to that position where the centre was at the lowest. The *locus* of

(Q 2)

the

the centre of gravity will, therefore, be the straight line QEC, drawn parallel to IDK, C being the centre of gravity of the rod in its initial position, and a given point, because CD is = AE, and, therefore, CEQ is given in position.

Now to investigate the nature of the curve, draw AS parallel to QC, and put $a = AE$, $b = AB$, cosine of the angle ASB = c , CS = x , and AS = y ; then, by similar triangles, $a : x :: b : \frac{bx}{a} = BS$, and, by trigonometry, $y^2 + \frac{b^2}{a^2} x^2 + 2 \frac{cb}{a} xy = b^2$,

which is manifestly an equation to a conic section. And because the cosine of an arc is always less than the radius $\frac{c^2 b^2}{a^2}$, or the square of half the co-efficient of xy is less than $\frac{b^2}{a^2}$ the co-efficient of x^2 ; and the squares x^2, y^2 have the same sign; therefore, the curve is an ellipse, whose centre is C. MacLaurin's Algebra, Part iii. Section 29.

XXVIII. QUESTION 58, by Mr. BAZLEY.

AC, CB are two rods or right lines connected at C by a moveable joint, and whilst the end C of the rod BC revolves round the fixed point B let the point A of the rod AC be carried along the right line AB passing through B; it is required to determine the equation and area of the curve described by P, a given point in AC.

SOLUTION, by Mr. LOWRY.

With the centre B (fig. 95, pl. 4.) and radius BC describe the circle bCH, then because the rod BC is of a given length, and revolves round the fixed pole B, the end C will move round in the circumference bCH, while the point A moves along the straight line aH. Let ab be taken equal to AC, and $ap = Hp' = AP$; then when C is at b or the rods are extended so as to coincide with aB, the point P will coincide with p , and when C comes to H, then P and p' will coincide; therefore, pp' are the points where the curve meets the axis AH. On AB drop the perpendiculars PQ, CE, and put $AC = a$, $AP = b$, $PC = c$, $BC = r$, $v = 3.14159$ &c. $pQ = x$, and $PQ = y$;

then, by similar triangles, $AP : PQ :: AC : CE = \frac{a}{b} y$, and,
therefore,

therefore, $AE^2 = a^2 - \frac{a^2}{b^2} y^2$. Again $a^2 : c^2 :: AE^2 : QE^2 = c^2 - \frac{c^2}{b^2} y^2$ or, $QE = \frac{b}{c} \sqrt{(b^2 - y^2)}$; and $BE^2 = BC^2 - CE^2 = r^2 - \frac{a^2}{b^2} y^2$ or, $BE = \frac{b}{a} \sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)}$. Now $Bp = r + c$, and, therefore, pQ , or $x = r + c - BE - QE = r + c - \frac{b}{a} \sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)} - \frac{b}{c} \sqrt{(b^2 - y^2)}$, which is the equation of the curve; and the fluxion of the area xy is

$$= \frac{\frac{a}{b} y^2 \dot{y}}{\sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)}} + \frac{\frac{c}{b} y^2 \dot{y}}{\sqrt{(b^2 - y^2)}} \text{ the fluent of which, by Article}$$

279 of Simpson's Fluxions, is

$$= \frac{r^2 b}{2a} A - \frac{ay}{2c} \sqrt{\left(\frac{r^2 b^2}{a^2} - y^2\right)} + \frac{bc}{2} B - \frac{c}{2b} y \sqrt{(b^2 - y^2)}, \text{ where}$$

A and B are the circular arcs, whose sines are $\frac{ay}{rb}$, and $\frac{y}{b}$ the radius being unity. Now when x becomes $=$ to pp' , y is $=$ a , and in this case the arcs A and B are each equal to half the circumference, and hence the whole area is $= \frac{vb}{2a} (r^2 + ac.)$

It is obvious from the manner of describing the curve, that it is of an oval form, having the parts on the different sides of pp' exactly equal and similar, and is a line of the 4th order in all cases except when the rods are of equal length, and then it becomes an ellipsis, whose centre is B ; for if a be wrote for r in the general equation of the curve, we obtain after proper reduction,

$$2(c+a)x - x^2 = \left(\frac{c+a}{b}\right)^2 y^2, \text{ which is an equation to an el-}$$

lipse whose semi-axes are $c+a$ and b .

When P has described a quadrant of this ellipse, BC will be perpendicular to the diameter bH , and the rods will coincide, so that during the remaining part of the revolution, P will describe the quadrant of a circle concentric to the given one; unless we suppose the rod AC to make a transit at B , and slide past the other rod, so that A may continue to move from B towards H , and then P will describe the remaining part of the semi-ellipse.

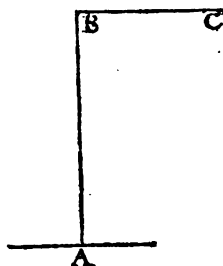
Moreover, if the rod BC , be longer than AC , it is obvious that BC can make only a partial revolution round B , so that A may keep in the line aH , for when AC becomes perpendicular

to that line, it is manifest that if BC was moved any nearer to H, the point A would necessarily quit aH, and move in some other line nearer to the circle, and, consequently, when BC has arrived at that position where AC is perpendicular to aH, it must move back again till it coincides with Bb; and then if bp'' be taken = PC, P will coincide with p'', and pp'' will be the axis of the curve.

Messrs. Dawes, and Bazley the proposer, also answered this question.

XXIX. QUESTION 59, by Sir TITUS TRIPPLICATE.

Suppose a given cylindrical tube or pipe ABC, whereof AB is perpendicular, and BC parallel to the horizon, to revolve round AB as an axis with a given angular velocity; the lower end A standing in water, whose surface is at A, and the whole tube being previously filled; it is required to determine the velocity with which the water will issue from the extremity C.



SOLUTION, by the Proposer.

Put

a = length of the arm BC in feet

b = height of AB in feet

t = time of a revolution in seconds

g = $16\frac{1}{2}$ feet, the space descended by heavy bodies in 1"

p = 3.1416 = periphery of a circle whose diam. is 1.

If we suppose the centrifugal force to be as the velocity generated thereby in 1"; then by the laws of central forces we shall have the centrifugal force of a particle x at x distance from B,

equal to $\left(\frac{2p}{t}\right)^2 x \div x = \frac{4p^2 x}{t^2}$, and the fluent $\frac{2p^2 a^2}{t^2}$ is

the centrifugal motive force of the column x ; whence, when $x = a$, the whole centrifugal motive force of the water in

the arm BC is $\frac{2p^2 a^2}{t^2}$. But by the laws of gravity $2ga : a ::$

$\frac{2p^2 a^2}{t^2} : \frac{p^2 a^2}{gt^2} =$ the length of the column whose pressure is

equal to the centrifugal force.

Now it is plain from the nature of the question that the column impelling the water out of the extremity of the arm BC is $\frac{p^2 a^2}{gt^2}$
— b

— b and that the column moved is $a + b$ which also expresses the space through which the water is accelerated. Hence the circumstances of the problem now appear to be the same as if a constant head of water of the height $\frac{p^2 a^2}{g t^2} - b$, impelled a column of water horizontally at the bottom of the length $a + b$. Therefore, the accelerative force is $\left(\frac{p^2 a^2}{g t^2} - b\right) \div (a + b)$, and the space or length of acceleration $= a + b$; and, consequently, the velocity generated $= (v = 2 \sqrt{f g s}) \ 2 g^{\frac{1}{2}} \sqrt{\left(\frac{p^2 a^2}{g t^2} - b\right)}$; or to adapt it to practical purposes, we may substitute for $2 g^{\frac{1}{2}}$ the quantity 5.3 as determined by the Experiments of l'Abbé Bossuet and others; and then the velocity is $5.3 \sqrt{\left(\frac{p^2 a^2}{g t^2} - b\right)}$.

It may also be added, that b must not exceed 33 feet, because a greater column cannot be supported by the atmosphere's pressure.

Solutions were also received from Messrs. Collins, Myers, and Pickering.

XXX. PRIZE QUESTION 60, by Mr. JAS. IVORY.

THEOREM. Let n be any integer number, and let m be another integer number less than n : then

$$0 = n^m - n \cdot (n-1)^m + n \cdot \frac{n-1}{2} \cdot (n-2)^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot (n-3)^m +$$

&c. the co-efficients being the same with the co-efficients of the binomial quantity $(a-b)^n$, and the progression, $n^m, (n-1)^m,$

$(n-2)^m$ to be continued till we arrive at 1^m . Required the investigation.

FIRST SOLUTION, by the Proposer, Mr. IVORY.

Let $A + Bx + Cx^2 + Dx^3 + \dots + Mx^n$ be an integral function of x , in which x^m is the highest power of x : Suppose the several terms of a series in arithmetical progression to be substituted

substituted for x , and denote the resulting values of the function by

$$y', y'', y''', y^{iv}, \&c.$$

Then, according to what is demonstrated by all the writers on finite differences, the several differences of the m th order in the progression $y', y'', y''', y^{iv}, \&c.$ will be all equal to one another; and of consequence the differences higher than the m th order will be equal to nothing. Now the general expression for the first of the n th order of differences is

$$y' - ny'' + n \cdot \frac{n-1}{2} \cdot y''' - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot y^{iv} + \&c.$$

which expression will be equal to nothing when m is less than n , according to what has just been shewn. It is evident that the Question is only a very particular case of the proposition now demonstrated,

The Question may likewise be demonstrated in the following manner. Let e denote the number of which the hyperbolic logarithm is unity; then

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c,$$

$$\text{therefore, } e^x - 1 = x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \&c;$$

For the sake of brevity write S for the series on the right hand of the equation and raise both sides to the n th power; then

$$S^n = e^{-n \cdot e} + n \cdot \frac{n-1}{2} \cdot e^{(n-2)} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot e^{(n-3)} + \&c.$$

develope each term of the expanded binomial into a series proceeding by the powers of x , and having collected all the co-efficients of the same power of x into one sum, the equation may be thus written,

$$S^n = A^{(0)} + A^{(1)} \times x + A^{(2)} \times x^2 + A^{(3)} \times x^3 \dots A^{(m)} \times x^m$$

where

$$A^{(0)} = 1 - n + n \cdot \frac{n-1}{2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \&c. = 0,$$

$$A^{(1)} = n - n \times (n-1) + n \cdot \frac{n-1}{2} \times (n-2) - \&c.$$

$$A^{(2)} = \frac{1}{2} \times \{ n^2 - n \times (n-1)^2 + n \cdot \frac{n-2}{2} \times (n-2)^2 - \&c.$$

$$A^{(3)} = \frac{1}{1 \cdot 2 \cdot 3} \times \{ n^3 - n \times (n-1)^3 + n \cdot \frac{n-1}{2} \times (n-2)^3 - \&c.$$

And

And in general,

$$A^{(m)} = \frac{1}{1.2.3.4 \dots m} \times \left\{ n^m - n \times (n-2)^m + n \cdot \frac{n-1}{2} \times (n-3)^m - \&c. \right\}$$

A^m denoting the co-efficient of x^m whatever number m may represent. Now the equation that has been found is independent of the value of x , which could not be the case unless the two series were identical, and the co-efficient of any power of x in the one series equal to the co-efficient of the same power of x in the other series. But all the powers of x , lower than the n th are evidently wanting in the series S^n ; therefore, the co-efficients of the same powers of x must be all evanescent in the other series, that is, we must have $A^{(m)} = 0$, so long as m is less than n ; or when m is less than n ,

$$n^m - n(n-1)^m + n \cdot \frac{n-1}{2} (n-2)^m - \&c. = 0.$$

COR. 1. The lowest power of x in the series S^n is x^n and its co-efficient is 1: therefore, we must likewise have $A^{(n)} = 1$, that is

$$1 = \frac{1}{1.2.3.4 \dots n} \times \left\{ n^n - n(n-1)^n + n \cdot \frac{n-1}{2} (n-2)^n - \&c. \right\}$$

$$\text{or } \left\{ n^n - n(n-1)^n + n \cdot \frac{n-1}{2} (n-2)^n - \&c. \right\} = 1.2.3.4 \dots n.$$

COR. 2. Let $O = 1 + \frac{x}{2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.$ then the co-efficient of x^m in the series O^n will be

$$\frac{1}{1.2.3.4 \dots (n+m)} \times \left\{ n^{n+m} - n(n-1)^{n+m} + n \cdot \frac{n-1}{2} (n-2)^{n+m} - \&c. \right\}$$

SECOND SOLUTION, by Mr. BAZLEY, Bolton.

If we take any residual $(z - 1)$ whose first term is any variable quantity, and the second term unity, and involve it by the binomial theorem, we have, as is well known,

$$z^n = nz^{n-1} + n \cdot \frac{n-1}{2} z^{n-2} - \&c. = (z-1)^n;$$

but in what follows it will be convenient, for brevity's sake, to

$$\text{put } A = n, B = n \cdot \frac{n-1}{2}, C = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}, \&c.$$

so that

$$z^n - Az^{n-1} + Bz^{n-2} - Cz^{n-3} \&c. = (z-1)^n.$$

Take fluxions in this expression and divide by $\dot{z}$, then

$$nz^{n-1} - (n-1)Az^{n-2} + B(n-2)z^{n-3} \&c. = n(z-1)^{n-1} \dots (1)$$

which when $z=1$, or $z-1=0$ gives

$$n - A(n-1) + B(n-2) - C(n-3) + \&c. = 0.$$

Multiply the expression (1) by z , take fluxions and divide by $\dot{z}$: then

$$n^2 z^{n-1} - A(n-1)^2 z^{n-2} + B(n-2)^2 z^{n-3} - C(n-3)^2 z^{n-4} =$$

$$n(z-1)^{n-1} + n.(n-1).z(z-1)^{n-2} \dots (2)$$

which when $z=1$ or $z-1=0$, gives

$$n^2 - A(n-1)^2 + B(n-2)^2 - C(n-3)^2 + \&c. = 0.$$

Multiply the expression (2) by z , take fluxions and divide by $\dot{z}$: then

$$n^3 z^{n-1} - A(n-1)^3 z^{n-2} + B(n-2)^3 z^{n-3} \&c. = n.(z-1)^{n-1}$$

$$+ 3n(n-1)(n-2).z(z-1)^{n-2} + n(n-1)(n-2).z^2(z-1)^{n-3} (3)$$

which when $z=1$ or $z-1=0$, gives

$$n^3 - A(n-1)^3 + B(n-2)^3 - C(n-3)^3 + \&c. = 0.$$

In like manner by continually repeating the same operations we shall at length arrive at a value of the expression

$$n^m z^{n-1} - A(n-1)^m z^{n-2} + B(n-2)^m z^{n-3} - \&c.$$

which value it is manifest will involve $(z-1)$ in each of its terms when m is less than n : therefore, putting $z=1$, or $z-1=0$, we shall have, when m is less than n

$$n^m - A(n-1)^m + B(n-2)^m - C(n-3)^m + \&c. = 0.$$

This method has the advantage of being applicable to the valuation of the expression $n^m - A(n-1)^m + B(n-2)^m - \&c.$ whether m is less than n or not.

$$\text{COR. } 1 - n + n \cdot \frac{n-1}{2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \&c. \text{ till } 1 = 0.$$

THIRD SOLUTION, by Mr. LOWRY.

In the series $en^m, (n-1)^m, (n-2)^m, \&c.$ where the quantities $n, n-1, n-2, \&c.$ are in arithmetical progression, the m th order of differences are equal (see Emerson's Differential Method,

Prop.

Prop. 4); and, therefore, all the orders of differences higher than the n th must be equal to 0: but by the question n is greater than m , therefore, the n th order of differences in the above series must be equal to 0. Now (by Prop. 1. *ibid*) the given series

$$n^m - n \cdot (n-1)^m + n \frac{n-1}{2} \cdot (n-2)^m - n \frac{n-1}{2} \cdot \frac{n-2}{3} (n-3)^m + \&c. \text{ is}$$

the first term of the n th order of differences in the series $n^m + (n-1)^m + (n-2)^m, \&c.$ and which is shewn above to be $=$ to 0, therefore,

$$0 = n^m - n \cdot (n-1)^m + n \frac{n-1}{2} \cdot (n-2)^m - n \frac{n-1}{2} \cdot \frac{n-2}{3} (n-3)^m + \&c.$$

FOURTH SOLUTION, by PORUS.

By the binomial $(1-1)^n$ is $= 1 - n + n \frac{n-1}{2} - n \frac{n-1}{2} \frac{n-2}{3} + \&c.$ and by Simpson's celebrated theorem (Mathema-

tical Differtations, page 62) if the terms of this series be respectively multiplied by any quantities $p, q, r, s, \&c.$ and the differences of these quantities be continually taken, and the first difference of the first order be denoted by D_1 , the first of the second order by D_2 , the first of the third order by D_3 , and in general the first of the n th order by D_n , the series

$$p - nq + n \frac{n-1}{2} \cdot r - n \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot s + \&c. \text{ thence arising will be } =$$

$$p(1-1)^n + n D_1 \cdot (1-1)^{n-1} + n \frac{n-1}{2} D_2 \cdot (1-1)^{n-2} \\ + n \frac{n-1}{2} \cdot \frac{n-2}{3} D_3 \cdot (1-1)^{n-3} + \&c.$$

Now it is manifest that all the terms of this latter series vanish except the one, in which the index of $(1-1)$ is 0, for $(1-1)^x$ is equal to nothing in every case but one, *viz.* when $x = 0$, and then $(1-1) = 0^0 = 1$; consequently the series

$$p - nq, \&c. \text{ is } = n \frac{n-1}{2} \dots \frac{n-(n-1)}{n} \times D_n = D_n,$$

from which general theorem the proposed one is directly deducible. For if $p = n^m$, $q = (n-1)^m$, $r = (n-2)^m$, $s = (n-3)^m$ &c. it is well known, and might be easily demonstrated by means
 (R 2) of

of the binomial theorem, that all their differences of the m th order are equal to each other, and, consequently, all their differences of a higher order vanish; but m is less than n by the hypothesis, therefore, $D_n = 0$, and because $D_n = p - nq + \&c$, we have by substitution

$$0 = n^m - n \cdot (n-1)^m + n \cdot \frac{n-1}{2} (n-2)^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} (n-3)^m + \&c. \dots \dots \dots Q. E. D.$$

SCHOLIUM. From the general theorem, $p - nq + n \cdot \frac{n-1}{2} \cdot r - \&c. = D_n$, a number of very curious properties may be derived; but the subject is foreign to the present question.

The Medal for solving the Prize Question is decided in favour of Mr. BAZLEY, who will please to send for it to Mr. GLENDINNING'S.

NOTICES

RELATING TO MATHEMATICS.



I. Prize Question of the National Institute of France.

“ To give a Theory of the Perturbations of the Planet Pallas, discovered by Dr. OLBERS.”

The labours of Mathematicians have now so far succeeded in overcoming the difficulties that occurred in the celebrated Problem of the Three Bodies, as to be able to determine with the requisite exactness, the irregularities of the Moon, and of the old Planets, arising from their mutual Perturbations. In the case of all the old Planets, as well as in that of the Georgium Sidus discovered about 20 years ago by Dr. Herschell, the excentricities of the orbits are small, and the inclinations to the plane of the ecliptic are confined within narrow limits; circumstances very favourable to abridge the computation of their numerous inequalities. The Planet Pallas, lately discovered by Dr. OLBERS, of which the excentricity is greater than that of Mercury, the most excentric of all the old Planets, and the inclination no less than $35^{\circ} 37'$, or more than five times greater than that of any other Planet, offers a new case of the Problem of the Three Bodies to the consideration of Astronomers, for which the formulæ investigated for the old Planets can be of little use. The intention of the Mathematical Class of the Institute in proposing the present subject for its Prize, is to call the attention of Mathematicians and Astronomers to this new case of the Problem, and to invite them to develop and discuss the formulæ, in order to give them the degree of precision necessary in the actual circumstances of the new Planet. As the Planet has not been observed for a sufficient length of time to determine with accuracy the elements of the orbit, it is required that the competitors for the Prize, shall give the analytical coefficients independent of numerical values. Thus every new discovery in nature requires a correspondent improvement

ment in theory. It is not difficult to foresee that Mathematicians will be obliged to find out methods of resolving the Problem of Perturbations entirely new; and that the wants of Astronomy will not be completely satisfied until a solution be accomplished that shall be independent of the elements of the orbit, and that shall converge only by the powers of the disturbing force.

II. NEW PLANET.

On the first of September last Mr. Harding, at the Observatory at Lilienthal, near Bremen, discovered a moveable star, in the sign Pisces, in light and apparent magnitude similar to the new Planet Ceres, and moving westward with a retrograde motion and increasing south declination. This Star is now considered to be a new Planet, of the Class of Ceres and Pallas, and in the same region. It has the appearance of a star of the eighth magnitude. Its place, as observed by Dr. Olbers, on Sept. 8th. $8^h 11^m 20^s$ M. T. was

Right asc.	Declin. South
$1^h 29' 39''$	$0^h 47' 19''$.

the daily motion in A. R. $7' 16''$ or $31.7''$ in time retrograde,
the daily motion in Declin. $12' 34''$ South.

In the Philosophical Magazine for October a chart of the apparent path of this new Planet is given.

III. DEATH OF MECHAIN.

On September the 20th, 1804, died the celebrated French Astronomer M. Mechain. He was employed with M. Delambre in the measurement of the Meridian from Dunkirk to Barcelona, which they completed in 1798. In 1803 he set out to continue the Meridional measurement as far as the Balearian Isles; and he has fallen a victim to the labour of his undertaking, and to the unhealthiness of the coast of Valentia in Spain.

IV. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philosophical Transactions of the Royal Society of London.
Abridged by Charles Hutton, L.L.D. F.R.S. George Shaw,
M.D. F.R.S. F.L.S. Richard Pearson, M.D. F.A.S.

The

The original work, of which the present is an abridgement, extends to about 100 volumes, and contains a mass of information not to be equalled by the proceedings of any other Society in the world. A work thus made up of the labours of the most learned men in Britain, as well as many in other parts of Europe, for a century and a half past, must be of infinite importance to all who are desirous of becoming acquainted with the origin and present state of human knowledge. Nothing therefore could be more auspicious to the cause of science than the present undertaking, which proposes to bring within a reasonable compass all that is truly valuable in the original work. Four volumes of the abridgement have now appeared, and have fully justified the public expectation, a circumstance which will not appear remarkable when it is recollected that the execution of the work is confided to men of the highest eminence in the departments of Science they have engaged to superintend.

An Account of the Astronomical Discoveries of Kepler; including an Historical Review of the Systems which had successively prevailed before his Time. By Robert Small, D.D. F.R.S.E.

The Principles of Analytical Calculation. By Robert Woodhouse, A.M.

An Analysis of the Principles of Natural Philosophy. By Mathew Young, D.D. S.F. T.C.D. late Bishop of Clonfort.

A Supplement to a Treatise on the Construction and Properties of Arches, published in 1801. By George Atwood, Esq.

The Philosophical Transactions of the Royal Society of London for 1804. Part I.

A Collection of Mathematical Tables, for the use of Students; for the Practical Navigator, Geographer, and Surveyor; for Men of Business, &c. By Andrew Mackay, L.L.D. &c.

The Elements of Natural and Experimental Philosophy. By Tiberius Cavallo, F.R.S.

A Practical Treatise on Perspective, on the Principles of Dr. Brook Taylor. By Edward Edwards.

Elements of Science and Art: being a familiar Introduction to Natural Philosophy and Chemistry, together with their Application to a Variety of elegant and useful Arts. By John Imison.
Meteorological

Meteorological Observations and Essays. By John Dalton, Professor of Mathematics and Natural Philosophy, at the New College, Manchester.

The Complete Navigator ; or, An easy and familiar Guide to the Theory and Practice of Navigation ; with all the requisite Tables : illustrated with Engravings. By Andrew Mackay, L.L.D. F.R.S.

Elements of Natural Philosophy ; explaining the Laws and Principle of Attraction, Gravitation, Mechanics, Pneumatics, Hydrostatics, Hydraulics, Electricity, and Optics : with a general View of the solar System, adapted to public and private Instruction : illustrated with Engravings. By John Webster.

The Elements and Practice of Mensuration and Land Surveying ; with an Appendix, containing Rules for measuring Hay-Stacks, Marl-pits, and Canals ; with numerous Figures and Copper-plates. By Joseph Becket.

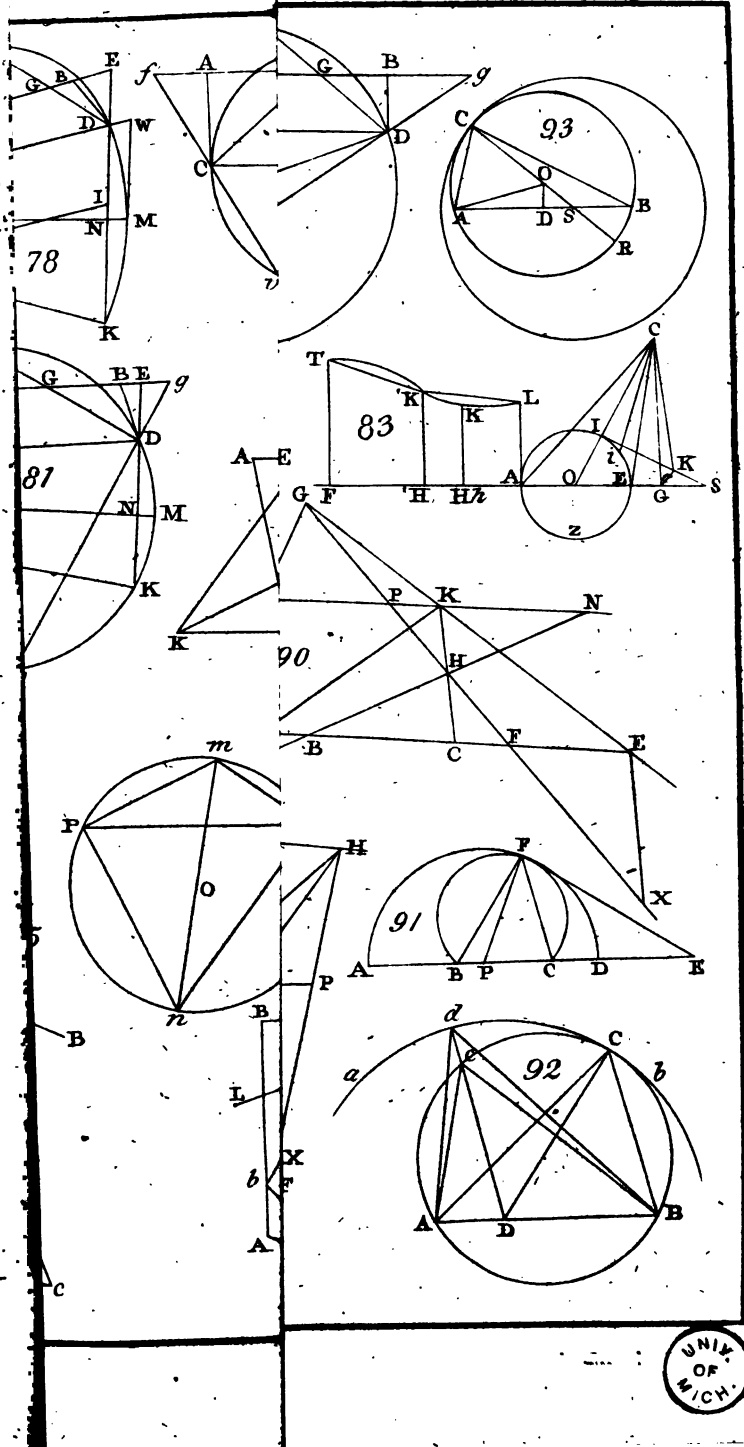
The Transactions of the Royal Irish Academy. Vol. IX.
Transactions of the American Philosophical Society. Vol. IV. and V.

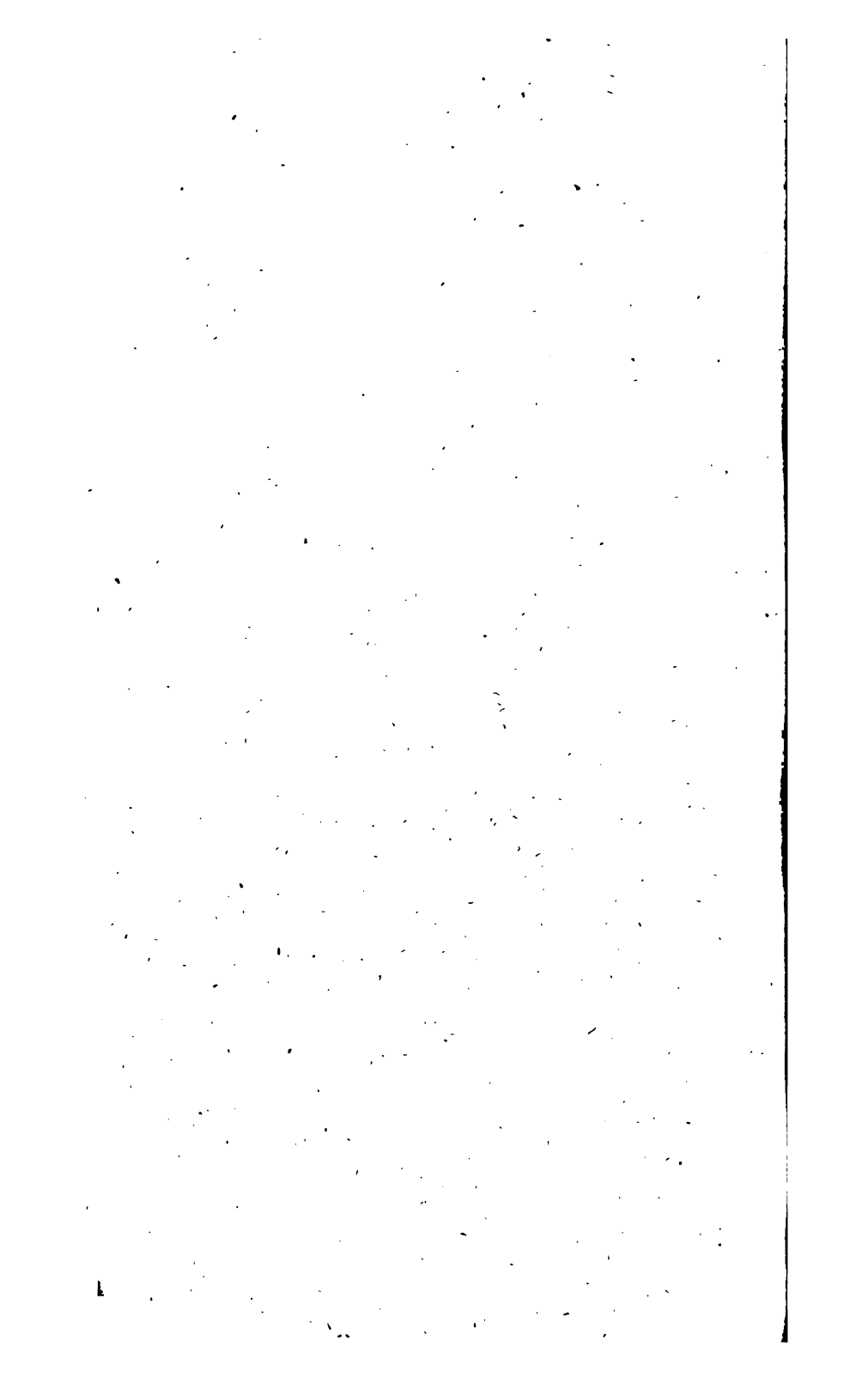
V. MATHEMATICAL WORKS IN THE PRESS.

The fifth volume of the Transactions of the Royal Society of Edinburgh will speedily be published.

Nearly ready for the press, and will be published early the ensuing year.

A Mathematical and Philosophical Glossary : exhibiting the derivations of words in the various branches of Mathematics and Philosophy from their original languages : with their accentuation, and accurate pronunciation according to grammatical propriety, and usage of the most esteemed Authors. The whole being so arranged, that the words appropriate to each subject may be found collected alphabetically under the general term, each concisely explained, and a reference for farther elucidation to the most explicit writers. By Henry Clarke, L.L.D. Professor of Experimental Philosophy in the Royal Military College, Marlow.





ARTICLE III.

*Solutions to Questions proposed in Number II.*I. QUESTION 61, *by Mr. THOMAS SQUIRE, Baldock.*

A tree growing upon the side of a hill which rises due North at an angle of 30° , had the upper part blown off 12 feet from the ground by a strong W. S. W. wind. It is required to find the height of the tree supposing it to stand perpendicular to the horizon, and the top (before the other part left the tree) to strike the ground 40 feet from its bottom ?

FIRST SOLUTION, *by Mr. THOMAS BOOLE.*

Let AEC (fig. 106, pl. 6.) represent the tree broken at E, AB an horizontal line in the direction of the foot of the hill, and C the point where the top of the tree falls upon the side of the hill. Conceive the line CD to be drawn perpendicular to the horizontal plane ABD, join AD, and draw DB perpendicular to AB; join also B, C, and A, C. Then there are three right angled triangles ADB, BDC, and ADC: the first being on the plane of the horizon, the second on the plane of the meridian, and the third on a plane in the direction of the wind. And there are given $\angle BAD = 22^\circ 30'$ $\angle DBC = 30^\circ$ $AE = 12$ and $AC = 40$, to find EC; and this will be easiest done by first finding the $\angle DAC$. Let AK be taken = to the radius in the tables, and draw KI, KL parallel to DB, DC respectively, and join IL, which will evidently be parallel to BC; then KI will represent the sine of $22^\circ 30'$ and rad. : tang. $30^\circ :: KI : KL = \text{fine } 22^\circ 30' \times \text{tang. } 30^\circ$. But LK is the tang. of $\angle LAK$ to the same radius AK; therefore, the log. tang. of $\angle LAK = (\text{DAC} =)$ the log. fine of $22^\circ 30' + \text{log. tang. } 30^\circ = \text{tang. } 12^\circ 27' 30''$ nearly. Hence $\angle EAC = 90^\circ - 12^\circ 27' 30'' = 77^\circ 32' 30''$ and the two sides AC and AE being given, EC is found = 39.204 , and consequently $12 + 39.204 = 51.204$ feet, nearly the height of the tree required.

SECOND SOLUTION, *by Mr. JOHN SMITH, Alton Park.*

Let ABC (fig. 107, pl. 6.) be a horizontal section, ABD a vertical section in a north direction, and ACE a vertical section in E. N. E. direction, or contrary to that of the wind, all passing from the foot A of the tree AFG through the hill. In the right angled triangles ABC, ABD are given the angle $BAC = 67^{\circ} 30'$ and the angle $BAD = 30^{\circ}$, whence, by trigonometry, AB, AC, and BD ($= CE$) are respectively as 1, 2.613126 and $.5773503$; and in the right angled triangle ACE the ratio of the sides AC to CE is given, viz. as 2.613126 to $.5773503$, therefore the angle CAE is $= 12^{\circ} 27' 32''$, the angle of the hill's declivity in the direction in which the upper part of the tree fell. Now in the triangle AFG, AF, AG, and the angle GAF are given, viz. $AF = 12$, $AG = 40$, and the angle $GAF (= \angle CAF - \angle CAE = 90^{\circ} - 12^{\circ} 27' 32'') = 77^{\circ} 32' 28''$; therefore FG is easily found $= 39.2032$ the length of the piece broken off. Hence $AF + FG = 51.2032$ feet, the whole length of the tree.

Solutions were also received from Messrs. Butterworth, Cavill, Collins, Dawes, Jones, Myers, N. R. D., Pickering, Ryley, and Squire.

II. QUESTION 62, *by Mr. JOHN CAVILL, Brighton.*

In any plane triangle it will be as the base is to the difference between the sum of the sides and base, so is the rectangle under half the perimeter and that part of the diameter of the circumscribing circle which is perpendicular to the base, and falls below it, to the area of the triangle. Required a demonstration?

SOLUTION, *by Mr. THOMAS MYERS.*

Let ACB (fig. 108, pl. 6.) be any plane triangle, FED the diameter of the circumscribing circle perpendicular to the base AB, and bisecting it in E. Join DC, DA, and let O be the centre, and OP the radius of the inscribed circle. Then the triangles

triangles DEA, OPC are similar, and therefore $AE : CP :: ED : OP$, or $AE : CP :: ED \times \text{half the perimeter} : OP \times \text{half the perimeter}$ (or the area of the triangle ACB). But it is well known that CP is equal to half the difference between the sum of the sides and the base, and, therefore, doubling the two first terms of the above proportion, AB is to the difference between the sum of the sides and the base, as $ED \times \text{half the perimeter}$ to the area of the triangle. Q. E. D.

Thus the proposition was demonstrated by Messrs. Bazley, Boole, Butterworth, Cavill, Collins, Dawes, Johnson, Jones, Mellor, Pickering, Roville and Ryley.

III. QUESTION 63, by HYPATIA.

A circular ring revolves on the inside of another exactly double its size; required the locus of a given point in its circumference?

FIRST SOLUTION, by Mr. S. JONES, *Ironcliff.*

In fig. 109, pl. 6. let P be the place where the given point in the lesser ring is in contact with the greater ring, and let C be the point of contact of the rings when the given point is in some other position as at p , and join OC, Op, and Pp. Then since every point of the arc pC is successively applied to the arc PC, it is evident that the arcs pC, PC are equal, and because the diameter of the circle APC is double the diameter of the circle OpC, it follows that the angle COp on the arc pC is equal to the $\angle COP$ at the centre on the arc PC; therefore the points O, p, P must be in the same straight line, and consequently the locus of p is the diameter AP.

COR. If pQ be a diameter of the lesser ring, the locus of Q will be the diameter ab, drawn perpendicular to the diameter AP.

SECOND SOLUTION, by Mr. G. WILLIAMS.

Let ACB (fig. 110, pl. 6.) represent the greater ring, and AO the less. Let AO revolve to the position OA'C, and the point A will move from A to A' along the diameter AOB. For suppose A, by the revolution of the ring AO, moves to any point a
(S 2) without

without the diameter AB, and draw the radius Ga meeting AB in n. Then since the circumference AC is equal to the circumference aC and the radius OA (by the question) equal to twice the radius GC. the angle CGa is equal to twice the angle COA, consequently, $Gn = GO = Ga$, which is absurd. Hence if the ring AO revolve along the femidiameter ACB, the point A will be always in AB, which is the locus required.

According to one or other of these Methods were the answers given by Messrs. Bazley, Boole, Butterworth, Cavill, Dawes, Johnson, N. R. D., Roville and W. Smith.

IV. QUESTION 64, by Mr. THOMAS HOPPER, Manchester.

Admitting the latitude of Manchester to be $53^{\circ} 25' N$. It is required to find the situation of two other places on the surface of the globe, such, that when it is noon at Manchester on the third day of March, 1805, the sun's altitude may be equal at all the three places; and the first place from Manchester may bear East-South-East, and the second place from the first West-North-West?

FIRST SOLUTION, by Mr. JOHN DAWES.

Fig. 111. pl. 6. Let the primitive circle represent the horizon; the right circle MN the meridian of Manchester; P the pole; Z the zenith; $\odot$ the sun on the Meridian; R and S the first and second places required; let $Z\odot = R\odot = S\odot$ = the given zenith distance of the sun, which on account of the difference of longitude of Manchester and Greenwich is $= 60^{\circ} 16' 46''$; let the $\angle \odot ZR$ be the given bearing of the first place from Manchester, and the $\angle PRS$ that of the first and second place, each $= 67^{\circ} 30'$; now the $\triangle \odot ZR$ (being isosceles) is known, and because ZR is known, therefore the $\triangle PZR$ is known: but because each of the two places bear under the same angle, i.e. $\angle \odot ZR = PRS = \odot RZ$, therefore $PRZ = \odot RS = 33^{\circ} 34' 42''$, wherefore the isosceles $\triangle R\odot S$ is given; and because RS is given $= 111^{\circ} 9' 43''$, therefore the $\triangle PRS$ is given: from these data,

the latitude of R the first place (comp. RP) $= 5^{\circ} 23' 51''$
 the diff. of long. of Manchester and R $= ZPR = 59 \quad 8 \quad 28$
 the latitude of S the second place (comp. PS) $= 18 \quad 45 \quad 00$
 the diff. of long. of Manchester and S $= ZPS = 55 \quad 22 \quad 38.$

SECOND

SECOND SOLUTION, *by* N. R. D.

Let M (fig. 112, pl. 6.) be the place on the sphere answering to Manchester, Z and Q the other two places where the sun's meridian altitude is the same as at Manchester, the place Z bearing E. S. E. from M, and the place Q bearing W. N. W. from Z.

Let P be the pole and S the sun in the meridian PM, and describe the arcs MZ, PZ, SZ, QZ, and QP. Then, because of the equal altitudes, the arcs SM, SZ, and SQ are equal; and the sun's declination on the third of March, 1805, being $6^{\circ} 51' 55''$ south, this added to the co-latitude of Manchester gives $SM = SZ = SQ = 60^{\circ} 16' 55''$. Again, as Z is E. S. E. of M, the angle SMZ is $67^{\circ} 30'$, and in the isosceles $\triangle SMZ$ the $\angle MSZ$ is found $= 79^{\circ} 45' 41.4''$. Also in the $\triangle PSZ$ we have $PS = 96^{\circ} 51' 55''$, $SZ = 60^{\circ} 16' 55''$, and $\angle MSZ = 79^{\circ} 45' 41.4''$, to find $PZ = 84^{\circ} 36' 21''$, and the angle $SPZ = 59^{\circ} 8' 33.8''$. Hence the place Z is in North latitude $5^{\circ} 23' 39''$, and in longitude $59^{\circ} 8' 33.8''$ E. from Manchester. Again, because Q is W. N. W. from Z, the $\angle PZQ = SMZ = \angle SZQ = 79^{\circ} 45' 41.4''$, hence $\angle PZS = 101^{\circ} 4' 29''$, and $\angle SZQ = SQZ = 33^{\circ} 34' 29''$; and in the isosceles $\triangle ZSQ$, the side QZ is found $= 111^{\circ} 9' 39''$. Also in the $\triangle QZP$ the side QZ, PZ, and the included $\angle PZQ$ are given to find $QP = 71^{\circ} 15' 5''$ and $\angle ZPQ = 115^{\circ} 31' 13''$; hence $\angle QPS = 55^{\circ} 22' 40''$, the longitude of Q west from Manchester, and the latitude $18^{\circ} 44' 55''$ North.

Messrs. Boole, Collins, and Hopper, the Proposer, also answered this Question.

V. QUESTION 65, *by* ARITHMETICUS.

THEOREM. Let N be any number whatever, and let $a =$ the difference of N, and the next greater square number, and $b =$ the difference of N, and the next less square number. Then $N - a \times b$ will be always a square number?

SOLUTION, *by* Mr. SAMUEL RYLEY.

Let $N + a = (x + 1)^2$, then, by the question,
 $N - b = x^2$;
 therefore $N = (x + 1)^2 - a = x^2 + b$,
 and $a = 2x + 1 - b$,
 or $-ab = -2bx - b + b^2$;

and,

and, therefore, $N - ab = x^2 + b - 2bx - b + b^2 = (x - b)^2$, which is manifestly a square number; or, if for x there be put its value, found from the equation $a = 2x + 1 - b$, we have

$$x - b = \frac{a - b - 1}{2}, \text{ and, therefore, } N - ab = \left(\frac{a - b - 1}{2} \right)^2$$

a square number, as before.

COR. If $N = ab$, x will be $= b$, and in this case $a = b + 1$ and $N = b^2 + b$; hence if N be composed of a number added to its square, it follows that $N - ab$ will always be $= 0$.

Thus if $N = 2^2 + 2 = 6$, then $b = 2$ and $a = 3$; and

$$N - ab = 6 - 3 \times 2 = 0.$$

Also, if $N = 3^2 + 3 = 12$, then $b = 3$ and $a = 4$; and

$$N - ab = 12 - 3 \times 4 = 0.$$

And again, if $N = 4^2 + 4 = 20$, then $b = 4$ and $a = 5$; and $N - ab = 20 - 5 \times 4 = 0$, &c. &c.

And thus it was answered by Messrs. Bazley, Boole, Butterworth, Cavill, Collins, Elliott, Limenus, Myers, N. R. D., Pickering, and W. Smith.

VI. QUESTION 66, by Mr. S. WADE.

At what time in latitude $50^\circ 31'$ (the sun being in the equinox) will the shadow of the stile on the plane of a direct west vertical sun-dial be a minimum?

SOLUTION, by Mr. JOHN DAWES, Birmingham.

When the shadow is a minimum it will fall on the six o'clock line upon the plane, which on account of refraction will not happen at the time the sun sets, nor when it apparently sets, but some where between those times; let EB (fig. 113, pl. 6.) be the true, and SC the apparent path of the sun; the refraction for this small interval of time may be supposed constant, therefore, when the sun is in the horizon at $\odot$, it will appear at S , and when S moves to C the shadow is the minimum required; therefore in the quadrantal $\triangle \odot SP$ are given $\odot S = 33'$ min. and $\angle P \odot S = 39^\circ 29'$ to find the $\angle \odot PS = 20', 59''$, which gives 1m 23s 56th. after six the time required.

Messrs. Boole, Collins, Elliott, and Pickering, sent Solutions to this Question.

VII.

VII. QUESTION 67, by Mr. BENJAMIN BEVAN.

In a plane triangle, let w be the centre of a circle passing through x , y , and z ; then will $Cw = Cc$, and be in the same right line; and $wx = wy = wz = 2R$, or the diameter of the circumscribing circle; where x , y , z , &c. represent the same points and lines as they denote in the Synopsis of Data, for the Construction of Triangles?

SOLUTION, by Mr. JOHN BUTTERWORTH, *Haggate*.

Let S , H and G (fig. 114, pl. 6.) be the points where the lines xc , yc , and zc meet the circumscribing circle, and draw the radii CS , CH , and CG ; also draw xw parallel to CS meeting cC produced in w , and join yw , zw ; then w is the centre of a circle passing through x , y and z . For it is now well known that $Sc = Sx$, therefore $Cc = Cw$ and $CH = Hy$, consequently yw is parallel to CH . But CH is $= CS$, therefore yw is $= wx = 2CS$. In like manner it is proved that $zw = xw = 2CS$; therefore w is the centre of a circle passing through the points x , y and z , and consequently the points c , C , w , are in a straight line, and $Cc = Cw$; and $yw = xw = zw = 2CS = 2R$. *Q. E. D.*

Thus nearly was the proposition demonstrated by Messrs. Boole, Dawes, and Johnson.

VIII. QUESTION 68, by ALCOR.

Let A and B be two given points, CD , CE , two straight lines given by position; let AF , BF be inflected to F any point in one of the given lines, and let BF meet CD in G , draw GH parallel to AB meeting AF in H , then H will be in a straight line given by position. Required the demonstration?

FIRST SOLUTION, by Mr. JOHN BUTTERWORTH.

Fig. 115, pl. 6. Draw BL parallel to CD , meeting CE in L , and join AL , CH ; then since HG is parallel to AB and BL parallel to CD , we have $AB : HG :: BF : FG :: BL : CG$, and by permutation $AB : BL :: HG : CG$; therefore the triangles ABL , HGC are similar, and, therefore, CH is parallel to AL ; But AB and BL are given lines; therefore, CG has to
CH

GH a given ratio, and HG makes a given angle with CG which is given by position; wherefore the locus of the point H is the straight line CH, and which is also given by position, because it passes through a given point C, and is parallel to AL, which is given by position.

SECOND SOLUTION, *by Mr. W. SIMPSON, Bolton.*

Fig. 115, pl. 6. Draw BL parallel to CD meeting CE in L, and join AL; draw CH parallel to AL, and it will be the locus sought.

Because BL is drawn from a given point parallel to a line CD, given in position and meeting CE, also given in position, in L, the point L will be given, and consequently AL will be given by position, and therefore CH will also be given by position; we have therefore only to prove that HG is parallel to AB. Now, by parallels, $AF : HF :: LF : CF :: BF : GF$, or $AF : HF :: BF : GF$, therefore HG is parallel to AB.

COR. If Dh be drawn parallel to AB meeting BG produced in h, the locus of the point h is also a straight line given by position.

Ingenious solutions were also received from Messrs. Bazley, Boole, Collins, Mellor, Myers, Pickering, Roville and W. Smith.

IX. QUESTION 69, *by* GEOMETRICUS.

Four straight lines are given by position, it is required to describe a triangle whose angles may be upon three of the lines, and whose sides may make given angles with the remaining line?

SOLUTION, *by* N. P. O. ROVILLE.

Let AB, CD, AE' (fig. 116, pl. 6.) be the three straight lines given by position, on which the angular points of the triangle are to be, and AH the other line with which the sides of the triangle are to make given angles. Draw any line EG to make one of the given angles with AH, and from the points G and E where it meets AB, AE, draw GF and EF to make with AH the other two angles; draw also AF to intersect CD in Q: Now let QP, QR be drawn parallel to EF, and FG respectively, and join PR, then PQR is the triangle required.

Because

Because QP is parallel to EF , and QR to FG , it is evident that PQ and QR make the given angles with AH ; and $AE : PE :: AF : QF :: AG : RG$; therefore PQ is parallel to EG ; and, consequently, PR makes also the given angle with AH .

Messrs. Bazley, Boole, Butterworth, Collins, and Simpson, likewise sent neat answers to this Question.

X. QUESTION 70, by the Rev. L. EVANS, Royal Academy, Woolwich.

It is required to bisect an arch of a circle by the description of circles only?

SOLUTION, by the Proposer.

Let BC (fig. 51, pl. 2.) be the given arc, and A the centre; with the centres B, C , and distances BA or CA , the radius of the circle describe two arcs AD, AE , each equal to the given arc BC . With the centres D, E , and distance $DC = BE$ describe two arcs to intersect at F ; then with the radius AF , and the same centres D, E , describe two other arcs which will intersect in the circumference at G , and the arc BC will be bisected at the point G .

DEMONSTRATION. The sides of the three triangles DBA, BAC, ACE , being respectively equal, we have the angle BCA equal to the angle CAE ; therefore BC is parallel to AE , and therefore $BAEC$ is a parallelogram. It is proved in the same manner that $BCAD$ is a parallelogram, and the diagonal AB is equal to the opposite sides BD, AC ; therefore $DC^2 (DF^2) = AB^2 + 2AD^2$. Again, because DA, AE , are parallel to BC , the points D, A, E , are in the same right line, and the triangles FAD, FAE having all their sides equal, have also the angles FAD, FAE equal, consequently they are right angles. In the same manner it is shewn that the angles DAG, EAG , are right angles. Therefore, $DF^2 = DA^2 + AF^2 = DA^2 + DG^2$, and $DG^2 = AD^2 + AG^2$; wherefore $DF^2 = AG^2 + 2AD^2$; consequently, $AB^2 = AG^2$ and $AB = AG$ the radius of the circle; therefore, G is in the circumference of the circle. And because AG is perpendicular to DE it will also be perpendicular to BC , which is parallel to DE , and therefore the arc BC is bisected in G .

Messrs. Boole, Cavill, Collins, Myers, and Pickering, answered this Question.

XI. QUESTION 71, by WHACUM.

Let AB be a given straight line, and P a given point without it, let CPD be drawn meeting AB in C, and let CP be to PD as AC to CB. Required the locus of the point D?

FIRST SOLUTION, by Mr. BAZLEY, Bolton.

Join AP (fig. 117, pl. 6.) and produce BA till AS = AB, draw SF parallel to AP, and PF parallel to AS cutting SF in F; also draw DOE parallel to AP meeting PF in O and SA in E; and lastly draw BG parallel to AP meeting PC produced in G. Then, by the similar triangles ACP and BCG, $AC : CB :: CP : CG$, and by hypothesis $AC : CB :: CP : DP$, therefore CG is equal to DP; and the triangles BCG, OPD being equal as well as similar; therefore $CB = OP = AE$, and since $AS = SB$, SE will be equal to AC. Again, since the triangles CPA, CDE are similar we have $CE : CA :: ED : AP$, therefore $EC \times AP = CA \times ED$, that is $SA \times AP = SE \times ED$; consequently the locus of D is an hyperbola described through the given point P to the asymptotes SF, SA.

If, (as in fig. 118, pl. 6,) the point D be taken on the other side of C, produce AP to Q, so that PQ may be equal to AP, and complete the rectangles AQSB, and APFB, and let SB produced meet PC produced in G, and draw DH parallel to AB and DO parallel to SB; then, as above, $CG = PD$, $CB = OP$, and $DH = AC$; also $GH = AP = BF = SF$, $GF = GH + HF = AP + HF = PQ + HF = SH$. Hence by the similar triangles GDH, GPF; $GH : HD :: GF : FP$; but $GH = AP = PQ = SF$, and $GF = SH$; therefore $SF : HD :: SH : FP$; consequently, $SF \times FP = SH \times HD$; and therefore the locus of the point D is an hyperbola described through the point P to the asymptotes QS, SB, which is the very same hyperbola as before, only differently situated.

COR. BP is a tangent at P. For when CP falls in with BP there is no intersection of the curve made by BP, and, consequently, BP touches it in P. This agrees with what is commonly shewn by writers on conics; for in fig. 117, $BA = AS$, and in fig. 118, $BF = FS$, and therefore by conic sections BP is a tangent at P.

SECOND

SECOND SOLUTION, *by Mr. W. SMITH.*

Produce AB till AS = AB, (fig. 117, pl. 6.) and draw SI parallel to AP; then a hyperbola described through the point P to the asymptotes SB, SH will be the locus required when the point D is above the point P, but if it is below the point P (as in fig. 118, pl. 6,) produce AP till PQ is = to AP, and draw QS parallel to AR meeting BS drawn parallel to AP in S; then a hyperbola described through the point P to the asymptotes SQ, SR will, in this case, be the locus required.

DEMONSTRATION. Draw the line PCD to any point in the curve, and let PF, DH, be drawn parallel to AB, the latter meeting AP produced in N; then, by the property of the hyperbola

$$HD \times HS = AP \times FP = AP \times HN, \text{ or}$$

$$HS = AP + PN : AP :: HN : HD, \text{ and by division}$$

$$PN : AP :: DN : HD; \text{ but by similar triangles}$$

$$PN : AP :: DN : AC : \text{ therefore } HD = AC;$$

and since HN = AB, DN is = to BC; and, consequently, AC : CB (DN) :: CP : PD. It is therefore manifest that the hyperbola PD is the locus required.

Ingenious Solutions were also received from Messrs. Boole, Elliott, Limenus, and Simpson.

XII. QUESTION 72, *by BOMBELLI.*

To divide $\frac{1}{4}$ into 5 parts such, that either part added to the cube of $\frac{1}{4}$ shall be a square?

FIRST SOLUTION, *by LIMENUS, Bruton, Somerset.*

Let the five parts into which $\frac{1}{4}$ is required to be divided be designated by $a, b, c, d,$ and $e,$ and let the square numbers to which those parts when severally increased by the cube of $\frac{1}{4}$ or $\frac{1}{64}$ are to be equal, be denoted by $\alpha^2, \beta^2, \gamma^2, \delta^2$ and $\epsilon^2,$ and we shall immediately have

$$a = \alpha^2 - \frac{1}{64}, \quad b = \beta^2 - \frac{1}{64}, \quad c = \gamma^2 - \frac{1}{64},$$

$$d = \delta^2 - \frac{1}{64}, \quad e = \epsilon^2 - \frac{1}{64}.$$

(T 2)

Wherefore

Wherefore having determined α, β, γ , &c. the parts a, b, c , &c. will become known, and the only restriction dependent on those equations is that α, β, γ , &c. be each greater than $\frac{1}{4}$, in order that all the parts may come out positive. The remaining condition is

that the sum of the parts be $\frac{1}{4}$, or that $\frac{1}{4} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$+ \epsilon^2 = \frac{5}{64}$, or $\frac{21}{64} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2$. It is therefore

necessary to divide the fraction $\frac{21}{64}$ into five squares, and to effect

such a division it will be requisite to assume some five numbers that possess that property, and then we shall be enabled to obtain a general solution of the problem.

The denominator of $\frac{21}{64}$ is itself a square, and the numerator 21

being but small, it will readily be perceived that the only ways in which it is capable of being formed into five integral squares are by the numbers 16, 4, 1, 0, 0; 9, 9, 1, 1, 1; and 9, 4, 4, 4, 0. And each of these sets will enable us to discover an infinite number of fractions possessing the required property; but that the same solution may be made applicable to them all, we will designate the roots of one of the sets by m, n, p, q, r , so that

$$m^2 + n^2 + p^2 + q^2 + r^2 = \frac{21}{64}.$$

Let v be some intermediate quantity, and l, k, λ, μ, ν , other component parts of the squares $\alpha^2, \beta^2, \gamma^2, \delta^2, \epsilon^2$, such that $\alpha = lv - m, \beta = kv - n, \gamma = \lambda v - p, \delta = \mu v - q$, and $\epsilon = \nu v - r$.

And by expanding the equation $\frac{21}{64} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2$,

and deducting from it that of $\frac{21}{64} = m^2 + n^2 + p^2 + q^2 + r^2$,

there will result

$$-2v(ml + nk + p\lambda + q\mu + vr) + v^3(l^2 + k^2 + \lambda^2 + \mu^2 + \nu^2) = 0,$$

$$\text{and thence } v = \frac{2ml + 2nk + 2p\lambda + 2q\mu + 2vr}{l^2 + k^2 + \lambda^2 + \mu^2 + \nu^2}.$$

In this formula the quantities l, k, λ, μ, ν are assumable at pleasure; and the quantities m, n, p, q, r , are to be determined by taking one of the sets of numbers before given. Having thus found a value of v , the corresponding ones of α, β, γ , &c. and thence the required parts a, b, c , &c. will become known.

By

By making use of the first set, and taking $l = 1$, $k = 1$, $\lambda = -1$, $\mu = 3$, and $\nu = 2$, we shall find $v = \frac{5}{8}$, and the five parts in this case will be $\frac{665}{4096}$, $\frac{161}{4096}$, $\frac{105}{4096}$, $\frac{57}{4096}$ and $\frac{36}{4096}$. Also, if we take $l = 3$, $k = 1$, $\lambda = -2$, $\mu = 3$, and $\nu = 2$, then $v = \frac{8}{9}$, and the five parts will come out $\frac{544}{5184}$, $\frac{495}{5184}$, $\frac{175}{5184}$, $\frac{63}{5184}$, and $\frac{19}{5184}$; and with the like facility may innumerable other solutions be derived.

SECOND SOLUTION, by Mr. CUNLIFFE, R. M. College.

Let the parts be denoted by w , v , x , y , and z , and by the question put

$$w + \frac{1}{64} = a^2, v + \frac{1}{64} = b^2, x + \frac{1}{64} = c^2, y + \frac{1}{64} = d^2 \text{ and } z + \frac{1}{64} = e^2.$$

By taking the sum of all the equations,

$$w + v + x + y + z + \frac{5}{64} = \frac{1}{4} + \frac{5}{64} = a^2 + b^2 + c^2 + d^2 + e^2.$$

Now put

$$\frac{3}{8} - mu = a, \frac{3}{8} - nu = b, \frac{1}{8} + pu = c, \frac{1}{8} + qu = d, \text{ and } \frac{1}{8} + ru = e;$$

Then

$$\frac{21}{64} = a^2 + b^2 + c^2 + d^2 + e^2.$$

$$\begin{aligned} &= \left(\frac{3}{8} - mu\right)^2 + \left(\frac{3}{8} - nu\right)^2 + \left(\frac{1}{8} + pu\right)^2 + \left(\frac{1}{8} + qu\right)^2 + \left(\frac{1}{8} + ru\right)^2 \\ &= \frac{21}{64} + u^2 \times (m^2 + n^2 + p^2 + q^2 + r^2) + \frac{u}{4} \times \{p + q + r - 3(m + n)\} \end{aligned}$$

From whence $u = \frac{3(m+n) - (p+q+r)}{4(m^2+n^2+p^2+q^2+r^2)}$, where m , n , p , q , and r , may be taken at pleasure; but with this restriction, that each of the quantities a , b , c , d , and e , shall exceed $\frac{1}{8}$.

Example

less, hence it is evident that the locus of the point C is a hyperbola of which F is a focus and EG the directrix.

COR. If upon AB any segment of a circle be described cutting the hyperbola which is the locus of the vertical angle in K, the part of the circumference AKB will be bisected in K. For join AK, BK, then the angle EBK being double the angle BAK, the circumference AK is double BK, therefore BK is one-third of BKA.

This question and its application to the trisection of an arch constitute Prop. 34, Book iv. of the Math. Collect. of Pappus.

SECOND SOLUTION, by Mr. W. SMITH, *Liverpool.*

Bisect the base AB in E, and draw the indefinite perpendicular EG; divide EB in V, so that $BV = 2EV$; with the focus B, vertex V, and directrix EG, describe a hyperbola, and it will be the locus required.

Draw AC, BC to any point in the curve, and draw CH perpendicular to EG, and produce it till $HI = HA$ and join CI. Then by the property of the hyperbola $EV : VB :: HC : CB$, but $BV = 2EV$, therefore $CB = 2CH = IC$: but since $AE = EB$, $IH = HC$, and EH perpendicular to AB and IC, it is obvious that AI is $\parallel$ to BC and the angle $BAI = ABC$; therefore IC is $\parallel$ to AE and the angle $ICA (BAC) = IAC$; and therefore the angle BAI or ABC is equal to twice the angle BAC.

This Question was also answered by Messrs. Bazley, Boole, Collins, Elliott, Johnson, Limenus, Myers, Pickering, and Simpson.

XIV. QUESTION 74, by Mr. GEORGE SANDERSON.

Determine the value of $\frac{\cos. \phi}{\cos. \psi}$, when $\frac{\sin. \phi}{\sin. \psi} = n$, a given number, and $\phi + \psi = \alpha$, a given arch.

FIRST SOLUTION, by SCOTICUS.

Suppose $\frac{\cos. \phi}{\cos. \psi} = x$, then $\cos. \phi = x \cos. \psi$ and $\sin. \phi = n \sin. \psi$, hence, putting P for $\cos. \phi \cos. \psi$, and Q for $\sin. \phi \sin. \psi$, we have $\cos^2. \phi = xP$, $\sin^2. \phi = nQ$, $nx \cos^2. \psi = nP$, $nx \sin^2. \psi = xQ$, therefore $xP + nQ = \cos^2. \phi + \sin^2. \phi = 1$, ... (1)
and $nP + xQ = nx (\cos^2. \psi + \sin^2. \psi) = nx$, ... (2)
The

The arithmetic of fines furnishes this other equation,

$$P - Q = \text{cof.}(\phi + \psi) = \text{cof.} \alpha \dots\dots\dots (3)$$

From the first and third of these equations we find

$$(n + x) P = 1 + n \text{ cof.} \alpha,$$

and from the second and third,

$$(n + x) P = x (n + \text{cof.} \alpha).$$

Therefore, $x (n + \text{cof.} \alpha) = 1 + n \text{ cof.} \alpha,$

$$\text{and } x = \frac{1 + n \text{ cof.} \alpha}{n + \text{cof.} \alpha}.$$

SECOND SOLUTION, by Mr. J. COLLINS, *Kenfington.*

Put $s =$ fine of the arch α , and c its cosine, the radius being denoted by r . Also put $x =$ the fine of the arch ϕ , and $y =$ fine of the arch ψ .

Then by known theorems.

$$s \times \text{cof.} \phi = ry + cx$$

$$s \times \text{cof.} \psi = rx + cy.$$

From whence

$$\frac{\text{cof.} \phi}{\text{cof.} \psi} = \frac{ry + cx}{rx + cy} = \frac{r + \frac{cx}{y}}{\frac{rx}{y} + c} = \frac{r + nc}{rn + c}. \quad Q. E. I.$$

Solutions were also received from Messrs. Bazley, Boole, Butterworth, Collins, Elliot, Myers, Pickering, Rville, and Simpson.

XV. QUESTION 75, by NAUTICUS.

Required the rectification of the line which is the projection of a great circle passing through a given point of the equator, and making with it a given angle in Mercator's Chart?

SOLUTION, by Mr. W. WALLACE, *R. M. College.*

Retaining the same notation as in the solution to question 46, in page 105 of the Repository, let us put farther z for the length of the curve line whose rectification is required. Then from what has been shewn in the solution referred to

$$\left\{ = \sqrt{(z^2 + j^2)} \right\} = \sqrt{\left(\phi^2 + \frac{a^2 \phi^2 \text{cof.}^2 \phi}{1 + a^2 \sin^2 \phi} \right)} = \frac{\sqrt{(1 + a^2) \phi}}{\sqrt{(1 + a^2 - a^2 \text{cof.}^2 \phi)}}.$$

Hence, dividing both numerator and denominator by $\sqrt{1+a^2}$, and putting e for the sine of the angle, which the great circle makes with the equator, that is, for $\frac{a}{\sqrt{1+a^2}}$, we find

$$\dot{z} = \frac{\dot{\phi}}{\sqrt{1-e^2 \cos^2 \phi}}, \text{ or, taking } \theta = 90^\circ - \phi,$$

$$\dot{z} = \frac{-\dot{\theta}}{\sqrt{1-e^2 \sin^2 \theta}}.$$

This fluxion has several remarkable properties, which have been considered by Euler, Lagrange, and Legendre, particularly by the last mentioned Author in his *Memoire sur les Transcendentes Elliptiques*. By a particular transformation its fluent admits of being expressed in a peculiar and very elegant manner. On this subject, see the last mentioned work, also a valuable paper, on the integration of certain differential expressions by Mr. WOODHOUSE of Cambridge, in the London Transactions for 1804. I shall only add, that the fluent may be expressed by two elliptic arches, the fluxion being the same as that which has occurred in the solution of Question 16, in page 38 of this Vol. but it is proper to observe, that in the 11th and 17th lines of that page, the word semi-conjugate seems, in each, to have been put by mistake for semi-transverse.

XVI. QUESTION 76, by THEODOSIUS.

Having given the base of a spherical triangle upon the surface of the sphere and its area; required the locus of its vertex?

SOLUTION, by Mr. LOWRY, R. M. College.

It has been shewn in several places that the locus required is a small circle of the sphere, equal and parallel to a small circle, described through the extremities of the given base; and the truth of this has generally been proved, by first shewing that spherical parallelograms on the same base, and between the same equal and parallel circles, are equal in area. This beautiful theorem is analogous to the thirty-fifth proposition of the first book of Euclid; and is a remarkable instance of the analogy which frequently subsists between spherical and rectilineal figures. In what follows I shall determine the locus from the properties of the supple-

mentary triangle, as this method is not only remarkably simple, but leads to the discovery of some other local theorems.

Let ABC (fig. 120, pl. 6.) be a spherical triangle described on the given base AB , and having its area, or which is the same, the sum of its three angles equal to a given quantity. About the angular points A, B, C , as poles, describe the supplementary triangle DEF , and continue the arcs DF, EF , till they meet as at G . Then it is well known that the sides DF, EF , and DE , are the supplements of the angles B, A, C , respectively, and the angle DFE the supplement of the base AB ; therefore when the base AB and the sum of the three angles A, B, C , are given, the angle DFE , and the sum of the three sides DF, EF, ED (or the perimeter of the triangle DFE) are given.

Conceive a small circle to touch the great circles DE, DG , and EG at Q, P , and H ; let O be the pole of this circle, and join OP, OH, OQ , and OC . Then, by the property of tangents, the arc $DQ = DP$, arc $EQ = EH$, and arc $PF = FH$; therefore PF and FH are each equal to half the perimeter of the triangle DEF , that is, equal to a given arc; and therefore the points H, P , are given, and of consequence the pole O , and the radius OQ (OH or OP) are given.

Also because C is the pole of the arc DE , it is evident that the locus of the point C will be the locus of the pole of the arc DE , and since DE touches the circle DQH at Q , the arc OQ is perpendicular to DE ; and, consequently, OQ produced will pass through the pole C , and QC will be a quadrant; therefore the arc OC , is equal to the arc $OQ + a$ quadrant, that is equal to a given arc; and O is a given point: therefore the locus of the point C is the circle CI , whose pole is O , and radius OC .

Because the arcs, HO, PO are perpendicular to GE, GD , they will pass through the poles A, B , of those circles, and therefore the arcs AO, BO , are each the complement of the arc OQ , or the supplement of the arc OC ; and, consequently, a circle described about the pole O to pass through the points A, B , will be equal and parallel to the circle CI , and will be the locus of that point on the sphere which is diametrically opposite to C .

Now since the great circle AB intersects one of the equal parallel circles at A and B , it must necessarily intersect the other parallel on the opposite side of the sphere. Let this be at m and n ; then, when C coincides with either of those points, Q will be either at H or I , and the three sides of the triangle ABC will coincide. Moreover, if C be beyond either of those points, Q will then fall between H and I on the side next to C , and the perimeter will be a variable quantity, which is contrary to the hypothesis, and therefore the locus is limited to that particular

ticular part of the circle mCn which is included between the points m and n .

We may infer from hence, that when the vertical angle and perimeter of a spherical triangle are given, the loci of the poles of the base will be two equal and parallel circles given in magnitude and position.

It is also obvious that the arcs Am, Bn , are each equal to the supplement of the base, and that the arc OF bisects the base AB at a .

Again, if a spherical triangle be described on the given base AB , so that the difference between the vertical angle and the sum of the angles at the base may be equal to a given angle, the locus of the vertex will be a circle.

Let the supplemental triangle, DEF (fig. 121, pl. 6.) be described as before, and let O be the pole of the circle PHQ inscribed in the triangle, and join OQ, OP, OH , and OC ; then, because the base and the difference between the vertical angle and the sum of the angles at the base are given, the angle DFE and the difference between the sum of the sides DF, EF , and the base DE will be given. Now the arc $DP = DQ$, arc $EH = EQ$, and arc $FP = FH$, therefore FH and FP are each equal to half the difference between the sum of the sides DF, EF , and the base DE , and therefore equal to a given arc. Hence it appears, in the same manner as before, that the point O and the arc OC are given; and, consequently, the locus of the point C is a circle whose pole is O , and radius OC .

The arcs OH, OP , produced will pass through the poles A and B respectively, and the arcs AO, BO , are each equal to the complement of the radius OQ , or equal to the arc OC , and therefore the locus required is the circle which circumscribes the triangle ACB .

If the arcs AC, BC , be produced till they meet as at G , the locus of the point G will be a circle equal and parallel to that which is the locus of the point C , and will therefore pass through the three points which are diametrically opposite to the points A, B, C . We may therefore infer, that if the vertical angle and the difference between the sum of the sides and base are given, the loci of the poles of the base are two given circles equal and parallel to each other.

The circle inscribed in the triangle DEF , and the circle circumscribing the triangle ABC have the same centre, and the radius of the one is the complement of the radius of the other.

That the circumscribing circle is the locus of the vertex when the difference between the vertical angle and the sum of the angles at the base is a constant quantity, may be proved in a manner different from the above, and suggests a very simple method

method of determining the locus, when the area is constant, without having recourse to the properties of the supplementary triangle.

Let ABC (fig. 122, pl. 6.) be the triangle whose base and area are given, and continue the sides CA, CB, till they meet at D, then it is evident that the angles D and C are equal, and $\angle CAB = 180^\circ - \angle DAB$, and $\angle CBA = 180^\circ - \angle DBA$; therefore the sum of the angles A, C, B, is equal to $360^\circ - (\angle DAB + \angle DBA) + \angle D =$ to a given quantity S, or $\angle DAB + \angle DBA - \angle D = 360^\circ - S$.

Let O be the pole of a circle described through the points, A, B, D, and draw the radii OA, OB, OD; then, by reason of the isosceles triangles, $\angle OBD = \angle ODB$, $\angle OAD = \angle ODA$; therefore $\angle D = \angle OBD + \angle OAD$; and, consequently, $\angle DAB + \angle DBA - \angle D = \angle OAB + \angle OBA = 2\angle OAB = 360^\circ - S$; therefore $\angle OAB = 180^\circ - \frac{1}{2}S =$ the supplement of half the given sum of the angles, and because AB is given, it is obvious that the pole O and the radius OD (OA, or OB) are given; therefore the locus of the point D is the circle DAB.

And since the point C is always at the distance of a semicircle from D, it is evident that the locus of C is a circle equal and parallel to that which is the locus of D.

The celebrated Legendre, in the notes to his *Eléments de Géométrie*, also determines the locus to be a circle: but by principles very different from the above. His investigation is as follows:

Let ABC (fig. 123, pl. 6.) be a spherical triangle described on the given base AB, and having a given surface $A + B + C - \pi = S$; π being $= 180^\circ$.

Let the arc IPK be drawn perpendicular to the base AB so as to bisect it at I, and take IP $=$ to a quadrant; then P will be the pole of the arc AB, and the arc PC, described through P and C, will be perpendicular to AB. Put $AB = c$, $ID = p$, $CD = q$, $AC = b$, and $BC = a$; then $AD = p + \frac{1}{2}c$ and $DB = p - \frac{1}{2}c$. And from the right angled triangles ACD, BCD, we have $\cos. a = \cos. q \cos. (p - \frac{1}{2}c)$ and $\cos. b = \cos. q \cos. (p + \frac{1}{2}c)$.

But it is well known that $\cot. \frac{1}{2}S = \frac{\cot. \frac{1}{2}a \cot. \frac{1}{2}b \cos. c}{\sin. C}$,

and writing for $\cos. c$, its value, $\frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$,

we have $\cot. \frac{1}{2}S = \frac{1 + \cos. a + \cos. b + \cos. c}{\sin. a \sin. b \sin. C}$,

and by substituting in this formula the values $\cos. a \cos. b = 2 \cos. q \cos.$

$g \cdot \cos. p \cdot \cos. \frac{1}{2}c, \text{ and } \cos. c = g \cos. \frac{1}{2}c, \sin. p \sin. C = \sin. c$
 $\sin. B = g \sin. \frac{1}{2}c \cos. \frac{1}{2}c \sin. B, \text{ we have}$

$$\cot. \frac{1}{2}S = \frac{\cos. \frac{1}{2}c + \cos. p \cos. q}{\sin. a \sin. \frac{1}{2}c \sin. B}.$$

But in the right angled triangle BCD, $\sin. a \sin. B = \sin. q$,
 therefore $\cot. \frac{1}{2}S = \frac{\cos. \frac{1}{2}c + \cos. p \cos. q}{\sin. \frac{1}{2}c \sin. q}$, or $\cos. p \cos. q =$

$\cot. \frac{1}{2}S \sin. \frac{1}{2}c \sin. q - \cos. \frac{1}{2}c$, which is the relation between p
 and q , and serves to determine the line which is the locus of all
 the points C.

On IP produced take $PK = x$, and let the arc joining KC =
 y , then in the triangle PKC we have $PC = \frac{1}{2}n - q$ and the
 angle KPC = $n - q$, and the side KC found by the proper
 formula is $\cos. KC = \cos. KPC \sin. PK \sin. PC - \sin. x \cos.$
 $q \cos. p$, in which substituting for $\cos. p \cos. q$ its value $\cot. \frac{1}{2}S$
 $\sin. \frac{1}{2}c \sin. q - \cos. \frac{1}{2}c$ we shall have $\cos. y = \sin. x \cos. \frac{1}{2}x +$
 $\sin. q (\cos. x - \sin. x \cot. \frac{1}{2}S \sin. \frac{1}{2}c)$; hence it appears that if
 we take $(\cos. x - \sin. x \cot. \frac{1}{2}S \sin. \frac{1}{2}c) = 0$, or which is the
 same, $\cot. x = \cot. \frac{1}{2}S \sin. \frac{1}{2}c$ we shall have $\cos. y = \sin. x \cos.$
 $\frac{1}{2}c$ and the value of y constant.

Therefore, if PK be taken such that its $\cot. = \cot. \frac{1}{2}S \sin. \frac{1}{2}c$,
 the vertices of all the triangles described on the same base c , and
 having the same surface S, are in the circumference of a small
 circle described from the point K, as a centre, with the polar dis-
 tance CK, such that $\cos. KC = \sin. PK \cos. \frac{1}{2}c$.

Legendre adds, that this beautiful theorem is due to L'evell,
 (See Tom. V. Part I. de *Nova Acta Petropolitana.*)

Messrs. Butterworth and Collins sent Solutions to this Question.

XVII. QUESTION 77, by Mr. JOHN WRIGHT, Norley.

Required the fluent of $x \sqrt{\frac{1+x^6}{(1+x^2)^3}}$, with the assist-
 ance of the conic sections?

SOLUTION, by Mr. CUNLIFFE, R. M. College.

In the first place,

$$x \sqrt{\frac{1+x^6}{(1+x^2)^3}} = \frac{x}{(1+x^2)^2} \sqrt{\frac{1+x^6}{1+x^2}} = \frac{x}{(1+x^2)^2} \sqrt{(1-x^2+x^4)}.$$

Put

Put $1 + x^2 = \frac{2x}{v}$; then $\frac{\dot{x}}{(1+x^2)^2} = \frac{v\dot{x}}{4x^2}$, and

$$1 - x^2 + x^4 = (1 + x^2)^2 - 3x^2 = \frac{4x^2}{v^2} - 3x^2 = \frac{4x^2}{v^2} \left(1 - \frac{3v^2}{4}\right);$$

taking the roots, $\sqrt{1 - x^2 + x^4} = \frac{2x}{v} \sqrt{1 - \frac{3v^2}{4}}$;

$$\text{therefore } \frac{\dot{x}}{(1+x^2)^2} \sqrt{1 - x^2 + x^4} = \frac{v\dot{x}}{2x} \sqrt{1 - \frac{3v^2}{4}}.$$

By taking the fluxion of the equation $1 + x^2 = \frac{2x}{v}$, we have

$$2x\dot{x} = \frac{2v\dot{x} - 2x\dot{v}}{v^2} \text{ which gives } v\dot{x} = \frac{x\dot{v}}{1 - vx}, \text{ or } \frac{v\dot{x}}{2x} = \frac{\frac{1}{2}\dot{v}}{1 - vx}.$$

Also from the same equation we get $x = \frac{1 - \sqrt{(1 - v^2)}}{v}$, or

$$vx = 1 - \sqrt{(1 - v^2)}; \text{ hence } 1 - vx = \sqrt{(1 - v^2)}, \text{ and}$$

$$\text{therefore } \frac{v\dot{x}}{2x} = \frac{\frac{1}{2}\dot{v}}{1 - vx} = \frac{\frac{1}{2}\dot{v}}{\sqrt{(1 - v^2)}}.$$

Consequently,

$$\dot{x} \sqrt{\frac{1+x^2}{(1+x^2)^2}} = \frac{\dot{x}}{(1+x^2)^2} \sqrt{1 - x^2 + x^4} = \frac{v\dot{x}}{2x} \sqrt{1 - \frac{3v^2}{4}} = \frac{1}{2} \dot{v} \sqrt{\frac{1 - 3v^2}{1 - v^2}},$$

which is at once recognised to be the fluxion of half the length of the arch of an ellipse intercepted between the conjugate

axis and a parallel ordinate at the distance of $v = \frac{2x}{1+x^2}$ from

the centre; the transverse axis of the ellipse being 2, and conjugate 1.

SECOND SOLUTION, by Mr. THOMAS BAZLEY.

Let the given expression be divided by $1 + x^2$ and it becomes

$$\dot{x} \sqrt{\frac{1 - x^2 + x^4}{(1 + x^2)^2}}, \text{ or } \frac{\dot{x}}{(1 + x^2)} \sqrt{1 - \frac{3x^2}{(1 + x^2)^2}}. \text{ Now to}$$

reduce this to a form more convenient for integration, put

$$1 + x^2 = \frac{2}{1 + z}; \text{ then } \frac{x}{1 + x^2} = \frac{-\dot{z}}{2\sqrt{(1 - z^2)}}, \text{ and by sub-}$$

stitution

situation the above expression becomes,
$$\frac{-\frac{z}{4} \sqrt{(1+z^2)}}{\sqrt{(1-z^2)}}$$

The fluent of which is the fourth part of the elliptic arc, reckoned from the extremity of the conjugate axis, whose ordinate to the transverse axe is z , semi-conjugate 1, and excentricity $\sqrt{3}$.

XVIII. QUESTION 78, by URSA MINOR.

It is required to demonstrate, independently of fluxions, that when an arch of a circle is small, twice the sine + the tangent = three times the arch nearly?

FIRST SOLUTION, by Mr. IVORY, R. M. College.

LEMMA.

Let the small arc ACB (fig. 124, pl. 6.) be bisected in C, and let AC be bisected in D: then the triangle ACB will be nearly equal to eight times the triangle ADC.

For, draw CO, DO to the centre of the circle, and let CO meet AB in m , and DO meet AC in n : then, because the arch BA is small the chord BA will be nearly double of the chord CA, and the chord CA nearly double of the chord DA: also, from the known property of the circle $Cm : Dn :: CA^2 : DA^2$ or nearly as 4:1; therefore Cm is nearly quadruple of Dn . Now the triangle BCA: triangle CDA :: $BA \times Cm : CA \times Dn :: 2 \times 4 : 1 \times 1 :: 8 : 1$ Q. E. D.

DEMONSTRATION OF THE QUESTION.

Let AB (fig. 125, pl. 6.) be the small arch, and AD its tangent: take the arch BE = arch AB, and join AE meeting BO drawn to the centre of the circle in C: bisect AB, BE in F and G, and join AF, FB, BG, GE: in like manner bisect the arches AF, FB, BG, GE, and inscribe a figure of eight sides in the segment ABE by drawing lines from the points of bisection to the points A, F, B, G, E: and suppose the inscribing of figures, by means of continual bisections, to be continued ad infinitum.

Then

Then, by the Lemma, the triangle EAB = 8 × triangle BFA; and the triangle BFA = 8 × triangle in the segment contained by the half of the arch FA; and so on, ad infinitum. But the circular segment ABE is composed of the $\Delta ABE + 2 \times \Delta AFB + 4 \times \Delta$ inscribed in the segment AF + 8 × Δ inscribed in the segment bounded by half the arch AF, and so on, ad infinitum: therefore the circular segment ABE

$$= \Delta ABE \times (1 + \frac{2}{8} + \frac{4}{8^2} + \frac{8}{8^3} + \frac{16}{8^4} + \&c. \text{ ad infinitum})$$

$= \frac{4}{3} \times \Delta ABE^*$. Therefore the semi-segment ABC = $\frac{4}{3} \times \Delta ABC$. But DO : OB :: OB : OC, therefore DB : BC :: OB : OC; and because the arch AB is small, therefore OB will differ little from OC; and, consequently, DB = BC nearly. Therefore $\Delta ACD = 2 \times \Delta ABC$: therefore semi-segment ACB = $\frac{2}{3} \times \Delta ACD$.

Therefore the outward space AFBD = $\frac{1}{3} \Delta ACD = \frac{1}{3}$ semi-segment ABC, but the outward space AFBD = $\frac{AO}{2} (AD - \text{arch AB})$: and the semi-segment ABC = $\frac{AO}{2} (\text{arch AB} - AC)$: therefore, arch AB - AC = $\frac{1}{3} (AD - \text{arch AB})$: consequently, arch AB = $\frac{2AC + AD}{3}$, or, to speak in strict geometrical language, the limit of the ratio of the small arch AB to $\frac{2AC + AD}{3}$ is the ratio of equality, and, the smaller the arch AB is, the more nearly equal will it be to the quantity $\frac{2AC + AD}{3}$. Q. E. I.

SECOND SOLUTION, by Mr. W. WALLACE, R. M. College.

Let ϕ denote a small arch of a circle, radius being unity.

$$\text{Then } \tan. \phi = \frac{\sin. \phi}{\sqrt{1 - \sin^2. \phi}} \left\{ \begin{array}{l} = \sin. \phi (1 - \sin^2. \phi)^{-\frac{1}{2}} \\ = \sin. \phi + \frac{1}{2} \sin^3. \phi + \frac{3}{8} \sin^5. \phi + \&c. \end{array} \right.$$

Now when the arch ϕ is small, the fifth and all higher powers of

* ARCHIMEDES has summed up this progression, in his Quadrature of the Parabola.

its sine will be very small, therefore in investigating an approximation to the arch they may be neglected, thus we have

$$\begin{aligned}\tan. \phi &= \sin. \phi + \frac{1}{2} \sin.^3 \phi \text{ nearly,} \\ \text{or, since } \sin.^3 \phi &= \frac{3}{4} \sin. \phi - \frac{1}{4} \sin. 3\phi, \\ \tan. \phi &= \frac{1}{2} \sin. \phi - \frac{1}{4} \sin. 3\phi \text{ nearly.}\end{aligned}$$

Let us assume $\sin. \phi = A\phi + B\phi^2 + C\phi^3 + D\phi^4 + E\phi^5 + \&c.$ where A, B, C, D, &c. denote numbers which are independent of any particular value of ϕ , then, since by the arithmetic of sines, when ϕ is negative its sine is also negative, we have

$$-\sin. \phi = -A\phi + B\phi^2 - C\phi^3 + D\phi^4 - E\phi^5 + \&c.$$

and, subtracting the latter equation from the former, and dividing both sides of the resulting equation by 2,

$$\sin. \phi = A\phi + C\phi^3 + E\phi^5 + \&c.$$

hence it appears that B, D, &c., the coefficients of the even powers of ϕ , are each necessarily = 0, and, therefore, that the series for the sine must consist only of the odd powers of the arch.

Let us divide both sides of the series by ϕ , and we have

$$\frac{\sin. \phi}{\phi} = A + C\phi^2 + E\phi^4 + \&c.$$

Now when $\phi = 0$ all the terms of the series vanish except A; hence it follows, that A expresses the limit of the ratio of the sine of an arch to the arch itself, when that arch is supposed continually diminished, but it may be rigidly demonstrated independent of the doctrine of fluxions, that the ratio in question is that of equality, therefore $A = 1$, and the series for the sine may be expressed thus,

$$\sin. \phi = \phi + B\phi^3 + C\phi^5 + \&c.$$

and substituting 3ϕ for ϕ

$$\sin. 3\phi = 3\phi + 27B\phi^3 + 243C\phi^5 + \&c.$$

so that, rejecting the fifth and all the higher powers of the arch, for the same reason as we have already rejected the like powers of the sine, we have

$$\sin. \phi = \phi + B\phi^3 \text{ nearly,}$$

$$\sin. 3\phi = 3\phi + 27B\phi^3 \text{ nearly,}$$

and since $\tan. \phi = \frac{1}{2} \sin. \phi - \frac{1}{8} \sin. 3\phi$ nearly,

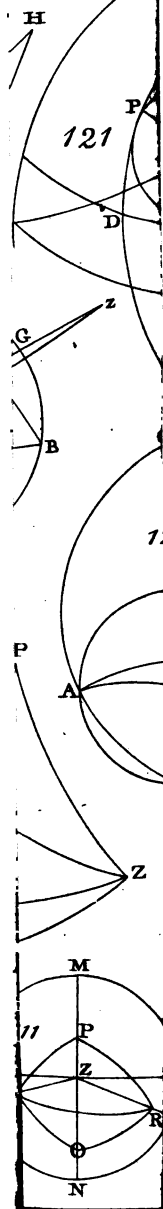
$$\tan. \phi = \phi - 2B\phi^3 \text{ nearly,}$$

hence, n being any number whatever,

$$n \sin. \phi + \tan. \phi = (n + 1) \phi + (n - 2) B\phi^3 \text{ nearly.}$$

Here if we suppose $n = 2$, the coefficient of ϕ^3 will be = 0, and thus $2 \sin. \phi + \tan. \phi = 3\phi$ nearly. And as in obtaining this approximation we have rejected no quantities of a lower order than the fifth power of the sine or of the arch, it may be inferred that the approximation is tolerably correct for small arches.

THIRD





THIRD SOLUTION, *by Mr. THOMAS BAZLEY.*

It is demonstrated by writers on trigonometry, that if a denotes the arc of a circle whose radius is unity, the corresponding

$$\text{fine is} \dots\dots = a - \frac{a^3}{6} + \frac{a^5}{120} - \&c. = s$$

$$\text{and } 2 \times \text{fine is} = 2a - \frac{a^3}{3} + \frac{a^5}{60} - \&c. = 2s$$

$$\text{and the tangent} = a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c. = t$$

and if the arc be very small all the terms after the second may be omitted as being extremely small in respect of the two first; hence by addition $2s + t = 3a$, which is the theorem in the question?

XIX. QUESTION 79, *by Mr. JOHN WRIGHT.*

Given the difference of the angles at the base, and the length of a line drawn from the vertex to the middle of the base to construct the triangle when the solid of the perimeter, the difference between the sum of the sides and the base, and the perpendicular let fall from the vertical angle upon the base is a maximum?

SOLUTION, *by Mr. J. JOHNSON, Birmingham.*

ANALYSIS. Suppose that ACB (fig. 126, pl. 7.) is the triangle required, IK the diameter of the circumscribing circle perpendicular to and bisecting AB at D; join KC, IC and DC, and draw CP parallel to AB meeting IK in P: then there is given DC, the angle CKI, (half the difference of the angles at the base), or its complement the angle CIK, and the solid

$$(AC + CB + AB)(AC + CB - AB) DP = \left\{ (AC + CB)^2 - AB^2 \right\}$$

$\times DP$, a maximum.

$$\text{Now } (AC + CB)^2 = 4ID \times PK \text{ and } AB^2 = 4ID \times DK, \\ \text{(X 2) and}$$

and their difference $= 4ID \times DP$, therefore $ID \times DP^2$ must be a maximum.

On DC let a semi-circle be described which will be the locus of the point P, also let a segment of a circle DCIL be described upon DC, to contain an angle equal to the angle DIC, or the complement of half the difference of the angles at the base, and this will be the locus of the point I. We have therefore only to draw the chord DI, so that $ID \times DP^2$ may be a maximum. Let DH be the diameter of the circle DICL, then it is evident that both DI and DP will increase till DI coincides with DH, and therefore the point I cannot be on the same side of DH that the point I is, but must either be at H or between H and C. Now if it was at H the angle DCI would be equal to a right angle, and if between H and C greater than a right angle; that is, the angle DCI would be equal to or greater than the angle ICK, of which it is only a part, which is impossible. Therefore no maximum can properly take place, and of course the problem is unlimited.

If it was required to draw DI so that $ID \times DP^2$ might be a maximum without having any regard to the triangle ABC. Let IC produced meet DB produced in R, and draw CQ parallel to DP; then it is manifest that DRC is a given angle, and therefore CQ ($= DP$) has to CR a given ratio, and DI to DR a given ratio; wherefore, when $ID \times DP^2$ is a maximum, $DR \cdot CR^2$ will be a maximum; wherefore in the triangle DCR, there are given the side DC, the opposite angle DRC and $DR \times CR^2$ a maximum, which is the same as question 842, Gentlemen's Diary.

Messrs. Simpson and W. Smith sent solutions to this question.

XX. QUESTION 80, by Dr. O'RIORDAN.

To this Question no satisfactory answer has been received.

XXI. QUESTION 81, by Mr. THOMAS BAZLEY, Bolton.

It is required to determine the figure of an orifice to be cut in the vertical side of a vessel or dam, whose water is always of the same constant height; of such a nature, that the quantity of water issuing below the shuttle shall be always as the height of the opening,

opening, and that when the whole orifice is open q cubic feet shall issue per second; the top of the orifice being n , and the bottom m feet, below the surface?

FIRST SOLUTION, by Mr. BAZLEY, the Proposer.

Let UR (fig. 127, pl. 7.) be the surface of the water, BoDERCB the orifice; oDSErIo the part below the shuttle; AKIS the axis bisecting all the ordinates oIr perpendicularly, the ordinates being parallel to the surface of the water UAR. Put $AK = n$, $AS = m$, $AI = x$, $oIr = y$, and $c = 5.3$ agreeably to the experiments of Bossut and others. Then the velocity

of the effluent water at I, is $cx^{\frac{1}{2}}$, and consequently the fluxion of the quantity issuing below the shuttle is, $-cyxx^{\frac{1}{2}} = -ax$ by the nature of the question, where a is constant; hence $cyx^{\frac{1}{2}} = a$,

or $y = \frac{a}{cx^{\frac{1}{2}}}$, the equation of the curve, which is of the hyperbolic

kind. Now to determine a , let $K - ax$ represent the correct fluent of $-ax$; and, per question, when $x = m$, $K - ax = K - am = 0$, so that $K = am$; and when $x = m$, $K - ax =$

$$am - an = q, \text{ whence } a = \frac{q}{m-n} = \frac{q}{KS}.$$

Moreover it appears that the general correct expression for the quantity of effluent water is, $am - ax = \frac{q}{m-n}(m-x) = \frac{q}{KS} \times SI$.

SECOND SOLUTION, by Mr. CUNLIFFE, R. M. College.

To make the consideration of this question as easy as possible, let us imagine the ordinates of the curve which bounds the orifice to be parallel to the horizon and perpendicular to the axis thereof. Also let us suppose the velocity of the effluent water at any depth, to be equal to that generated by gravity in an heavy body falling freely the same depth. Put $g = 16\frac{1}{2}$ feet, and let y denote the length of an ordinate at the depth x below the surface; also let l denote the length of the lowermost ordinate.

Then by the hypothesis $2\sqrt{gx}$ will express the velocity of the effluent water at the ordinate y , and $2\sqrt{gl}$ will express the velocity of the issuing water at the bottom of the orifice or at the ordinate l . And as the quantity of water issuing below the shuttle is to be always proportional to the height of the opening, it is

is evident that the quantity issuing at every ordinate in equal times must be the same.

Now the quantity of water issuing at any ordinate in the same time is proportional to the rectangle or product of the ordinate and the velocity of the efflux; therefore $2y\sqrt{gx} = 2l\sqrt{gm}$ or $y^2x = l^2m$, which is an equation expressing the nature of the curve which bounds the orifice.

It has already been observed, that the quantity of water issuing at every ordinate in equal times must be the same, and consequently the water issuing through the whole orifice in a second when open, will be the same as would issue in the same time through an horizontal rectangular hole whose sides are $2l$ and $m - n$ at the depth m below the surface.

Now the quantity of water which would issue through an horizontal rectangular hole whose sides are $2l$ and $m - n$ at the depth m below the surface of the water in a second of time will, by the known principles of hydrostatics, be expressed by $4l(m-n)\sqrt{gm}$, and this by the question is to be equal to q cubic feet, whence we get

$$l = \frac{q}{4(m-n)\sqrt{mg}} \text{ and } y^2x = l^2m = \frac{q^2}{16(m-n)^2g}.$$

Mr. Elliott sent a Solution to this question,

XXII. QUESTION 82, by Mr. W. WALLACE.

Given $\begin{cases} \text{cof. } \phi + \text{cof. } \psi = a \\ \text{cof. } 5\phi + \text{cof. } 5\psi = b \end{cases}$ To find $\text{cof. } \phi$ and $\text{cof. } \psi$?

FIRST SOLUTION, by the Proposer.

Let $\text{cof. } \phi$ and $\text{cof. } \psi$ be denoted by x and y respectively, and let $\text{rad.} = 1$, then by the arithmetic of sines

$$\text{cof. } 5\phi = 16x^5 - 20x^3 + 5x,$$

$$\text{cof. } 5\psi = 16y^5 - 20y^3 + 5y;$$

and therefore,

$$\text{cof. } 5\phi + \text{cof. } 5\psi = 16(x^5 + y^5) - 20(x^3 + y^3) + 5(x + y).$$

Now if we put a for $x + y$ and v for xy , it is evident that

$$x^5 + y^5 = a^5 - 5v(x^3 + y^3) - 10av^2,$$

$$x^3 + y^3 = a^3 - 3av.$$

Let

Let these values of $x^5 + y^5$, $x^3 + y^3$ and $x + y$ be substituted in the above expression for $\text{cof. } 5\phi + \text{cof. } 5\psi$, which by hypothesis is equal to a given number b , and we have

$$b = 16a^5 - 20a^3 + 5a + (60a - 80a^3)v + 80av^2.$$

The value of v , or of xy , may now be found by the resolution of this quadratic equation, and hence, and from the equation $x + y = a$, the values of x and y , or of $\text{cof. } \phi$ and $\text{cof. } \psi$, may be found by the resolution of another quadratic equation.

Note. In like manner if besides $\text{cof. } \phi + \text{cof. } \psi$ there be given any one of these $\text{cof. } 2\phi + \text{cof. } 2\psi$, $\text{cof. } 3\phi + \text{cof. } 3\psi$, $\text{cof. } 4\phi + \text{cof. } 4\psi$, the values of $\text{cof. } \phi$ and $\text{cof. } \psi$ may also be found by the resolution of quadratic equations: But if we advance to $\text{cof. } 6\phi + \text{cof. } 6\psi$, the equation to be resolved is a cubic.

SECOND SOLUTION, by Mr. G. WILLIAMS, Caermarthen.

Put $\frac{1}{2}y = \text{cof. } \phi$, and $\frac{1}{2}v = \text{cof. } \psi$. Then, by the question $y + v = 2a$, and by Simpson's Algebra, p. 200, we have

$$\text{cof. } 5\phi = \frac{y^5 - 5y^3 + 5y}{2} \text{ and } \text{cof. } 5\psi = \frac{v^5 - 5v^3 + 5v}{2}.$$

$$\text{Therefore } y^5 + v^5 - 5y^3 - 5v^3 + 5y + 5v = 2b, \\ \text{or because } 5y + 5v = 10a,$$

$$y^5 + v^5 - 5y^3 - 5v^3 = 2b - 10a.$$

Assume $y - v = 2z$, then $y = a + z$, and $v = a - z$:

$$\text{Therefore } y^5 + v^5 = (a+z)^5 + (a-z)^5 = 2a^5 + 20a^3z^2 + 10a^2z^4$$

$$\text{And } 5y^3 + 5v^3 = 5(a+z)^3 + 5(a-z)^3 = 10a^3 + 30az^2.$$

Hence, by transposition, &c.

$$z^4 + (2a^2 - 3)z^2 = a^2 - \frac{1}{5}a^4 + \frac{b}{5a} - 1, \text{ from which}$$

$$z = \sqrt{\left\{ \sqrt{(p^2 + q)} - p \right\}}, \text{ where } p = \frac{2a^2 - 3}{2}, \text{ and } q =$$

$$a^2 - \frac{1}{5}a^4 + \frac{b}{5a} - 1.$$

$$\text{Then } \text{cof. } \phi = \frac{1}{2}y = \frac{a+z}{2} \text{ and } \text{cof. } \psi = \frac{1}{2}v = \frac{a-z}{2}.$$

This Question was answered by Messrs. Collins, Myers, and Pickering.

XXIII. QUESTION 83, by Mr. IVORY, R. M. College.

Let A and B be two points without a circle, of which O is the centre, and let $BO \times OC = \text{square of radius}$, and $AO \times OD = \text{square of radius}$; and let BD and AC meet in E. Then, if the chord FG be drawn through E, and AF, FB, BG, AG be joined, $AF \times FB : AG \times GB :: FE : EG$. Required the demonstration?

FIRST SOLUTION, by Mr. JOHN JOHNSON, Birmingham.

Fig. 128, pl. 7. Draw the radii OF, OG, and produce OE to H so that $OE \times OH$ may be equal to the square of the radius OF, and join FH: then because $OE : OF :: OF : OH$, and the angle at O common to the triangles FOH, FOE, they are similar, and consequently the angle OFG = OHF; but the angle OFG is = to OGF (because OF = OG); therefore the angle OHF = OGF, and the points O, G, H, F, are in a circle; wherefore $FE \cdot EG = OE \cdot EH$. Again, because $OE \cdot OH = OD \cdot OA = OC \cdot OB$ (each being equal to the square of the radius), the points A, D, E, H, are in a circle, and so are also the points A, D, C, B; therefore the $\angle AHO = ODE = OCA$, and consequently the points A, H, C, O, are in a circle; and therefore $AE \cdot EC = OE \cdot EH$: but it has been shewn above that $FE \cdot EG = OE \cdot EH$; therefore $AE \cdot EC = FE \cdot EG$, and the points A, F, C, G, are in a circle.

Now since $OC \cdot CB = \text{the square of the radius}$, it is well known as a proposition of the *Loci Plani* that

$$BF : CF :: BG : CG, \text{ or } BF : BG :: CF : CG;$$

$$\text{Therefore } AF \cdot BF : BG \cdot AG :: CF \cdot AF : CG \cdot AG.$$

But by the property of the circle and inscribed trapezium

$$CF \cdot AF : CG \cdot AG :: EF : EG;$$

$$\text{therefore } AF \cdot BF : AG \cdot BG :: EF : EG.$$

Q. E. D.

$$\text{COR. 1. } AF \cdot BF : AG \cdot BG :: FH : HG.$$

$$\text{COR. 2. } CF \cdot DF : DG \cdot CG :: EF : EG :: FH : HG.$$

SECOND SOLUTION, by Mr. WM. SIMPSON, Bolton.

Fig. 128, pl. 7. Join FC, CG, and draw FP, GQ perpendicular to AC. Let OE be produced to meet the circle in I, and

and draw AH meeting OE in H, so that the angle EAH may be equal to the angle EOC.

Then because the rectangles $CO \cdot OB$, $DO \cdot OA$ are each equal to the square of the radius, the points C, D, A, B, are in a circle, and therefore the angles at C and D are equal, and consequently the angle $AHE = ECO = EDO$, and therefore the points D, E, H, A, are in a circle.

Wherefore $DO \cdot OA$, or $(OI)^2 = EO \cdot OH = (OE)^2 + OE \cdot EH$; but the points A, H, C, O, are also in a circle; therefore $OE \cdot EH = AE \cdot EC$, and $(OI)^2 = (OE)^2 + AE \cdot EC$, or $(OI)^2 - (OE)^2 = AE \cdot EC$. But $(OI)^2 - (OE)^2$ is $= EK \cdot EI$, wherefore $AE \cdot EC = EK \cdot EG$, and the points A, F, C, G, are in a circle.

Now because $OC \cdot OB$ is equal to the square of the radius $FB : FC :: BG : CG$ (Playfair's Euclid, B. VI. P. F.),

and by permutation $FB : BG :: FC : CG$; therefore

$$AF \cdot FB : AG \cdot BG :: AF \cdot FC : AG \cdot CG.$$

But it has been shewn that the points A, F, C, G, are in a circle, and (Prop. C. Ibid) the rectangles under the diameter of this circle and the perpendiculars FP, CQ, are respectively equal to the rectangles $AF \cdot CF$, $AG \cdot CG$; therefore it is evident that

$$AF \cdot FB : AG \cdot BG (:: FP : CQ) :: FE : EG \text{ (by similar triangles).}$$

Ingenious demonstrations were also received from Messrs. Bazley, Limenus, and W. Smith.

XXIV. QUESTION 84, by Mr. BENJAMIN BEVAN.

To this Question no answer has been received.

XXV. QUESTION 85, by Dr. O'RIORDAN.

To this Question no satisfactory answer has been received.

XXVI. QUESTION 86, by SCOTICUS.

If four straight lines intersect each other, and form four triangles, circles described about each of these triangles will pass through one and the same point. Required the demonstration?

FIRST SOLUTION, by Mr. GEORGE PICKERING.

Let AB, AC, GH, GC, (fig. 129, pl. 7.) be the four lines forming the four triangles AHE, ABC, GBE, and GHC. Let the circles AHE, EBG, circumscribing the triangles AHE, EBG intersect in P. Then the circles ACB, HCG described about the remaining triangles ABC, GHC, will also pass through the points P. Join PA, PH, PE, PB, and PG. Because the point P, E, B, G, are in a circle, the angle PGB is equal to the angle PEA ; but the angle PEA is equal to the angle PHA ; therefore the angle PGB or PGC is equal to the angle PHA ; hence the points P, H, C, G, are in the circumference of the same circle.

Again the angle PAH is equal to the angle PEG , that is equal to the angle PBG ; consequently the points P, A, C, B, are in a circle. Hence all the circles pass through the same point P.

SECOND SOLUTION, by Mr. JOHN CAVILL, Beighton.

Let AF, AR, BF, and RC, (fig. 130, pl. 7.) be the four lines intersecting each other in the points A, B, F, R, E, and G, and forming thereby the four triangles ABF, ARE, BGR, and EGF; about the triangles BGR and EGF, let circles be described intersecting again in D, draw ED, FD, BD, and RD; then the angle $\text{EGF} = \text{EDF} = \text{BGR} = \text{DBR}$, that is the angle $\text{EDF} = \text{BDR}$, add the angle EDB to each, and we have the angle $\text{FDB} = \text{EDR}$; also the angle $\text{DEG} = \text{DFG}$, consequently the triangles EDR , FDB are similar, and $\text{ED} : \text{FD} :: \text{DR} : \text{DB}$; therefore the triangles BDR , FDC are similar, and the angle $\text{DRB} = \text{FED}$.

Again the angle $\text{AED} = \text{EFD} + \text{EDF}$ and $\text{AED} + \text{ARD} = \text{EFD} + \text{EDF} + \text{FED} = \text{two right angles}$; consequently a circle will pass through the points A, E, D, R. And since the angle $\text{EDR} = \text{FDB}$, and the angle $\text{EAR} + \text{EDR} (= \text{two right angles}) = \text{FAB} + \text{FDB}$, therefore a circle will pass through the points A, F, D, B. Therefore circles described about the four triangles AFB, AER, BGR, and FEG, will all pass through the same point D, which was to be demonstrated.

Messrs. Bazley, Collins, Myers, Roville, Simpson, W. Smith, and Williams, answered it.

XXVII. QUESTION 87, *by Mr. JAMES NORTH.*

From the three angular points of a given plane triangle to draw three right lines meeting in the circumference of a given circle so as to have a given ratio to one another, the plane of the circle being perpendicular to the plane of the triangle.

SOLUTION, *by Mr. LOWRY, R. M. College.*

Let ABC (fig. 131, pl. 7.) be the given triangle, and P a point above or below the plane ABC, such, that the lines AP, BP, CP, drawn from that point to the angular points of the triangle may have a given ratio.

Take the points E, F, and G, H, such, that AE may be to EB and AF to BF in the given ratio of AP to BP; and that AG may be to GC and AH to CH in the given ratio of AP to CP.

Let two circles be described on the diameters EF, GH, intersecting each other in the points D, *d*, and join AD, *Ad*, BD, *Bd*, CD and *Cd*; then it is well known (Apol. Loci Plani, Lib. 2, Prop. 2.) that if two lines be drawn from the points A, B, to any point in the circumference of the circle described on EF, they will have the same ratio as AE to EB, or AP to BP; and also that two lines drawn from the points A, C, to any point in the circumference of the circle described on GH, will have the same ratio as AG to GC, or AP to CP; therefore it is obvious that the lines AD, BD, CD, and *Ad*, *Bd*, *Cd*, have the same ratio as the lines AP, BP, CP.

Now if the points D, *d*, coincide, or the circles only touch each other, it is manifest that the point of contact will be the only point from which lines can be drawn to the points A, B, C, so as to have the given ratio. But when the circles intersect each other as at D and *d*, let a circle be described on the diameter D*d*, and have its plane perpendicular to the plane ABC, then the point required may be any where in the circumference of this circle.

For let P be any point in the circumference, O the centre, and I the centre of the circle EDF. Join DI, OI, PI, and PO; then because D*d* is bisected in O, and the plane DP*d* is perpendicular to ACB, OI, and OP, are both perpendicular to OD; and OP is equal to OD, and OI common to both the triangles IOD, IOP; therefore, IP is equal to ID = IE = IF; and, consequently, the point P is in the circumference of a circle

(Y 2)

whose

whose diameter is EF ; and, therefore, (*Ibid.*) AP will be to BP in the given ratio of AE to EB ; and, in like manner, it is shewn that AP is to CP in the given ratio of AG to GC ; therefore the lines drawn from the angular points A, B, C , to any point whatever in the circumference DPd will be in the given ratio.

It therefore appears that the position of the point P can only be determined when the circumference of the circle DPd happens to meet the circumference of the circle which is given by position, for in any other case the problem will be impossible.

If the distances AP, BP, CP are equal, the locus will be a straight line perpendicular to the plane of the triangle ABC , and passing through the centre of its circumscribing circle.

What has been said will appear very evident if we conceive the two circles EDF, GDH , to revolve round their diameters so as to generate two spherical superficies, for the mutual intersection of those superficies will be the circle DPd , which is the locus of the point P .

We may also from hence determine the position of a point such, that if lines be drawn from it to the four angular points of a given trapezium, they may have a given ratio.

XXVIII. QUESTION 88, by Mr. JAMES IVORY.

- Let there be an ellipse of which AB is the greater, and ED the less axis and centre C . Take CH in either axis equal to the sum or difference of the two semi-axis CB and CD , and upon CH let a circle be described; through B , the extremity of the axis, draw a straight line to cut the circle in F and G , and through F and G draw two diameters of the ellipse PC and QC . Then, the line FG will be equal to the sum of the semi-diameters CQ and CP in the one case, and to their difference in the other; and if any line LK , equal to FG , be interposed between the diameters CP and CQ , and in LK (or in LK produced beyond the point K when CH is equal to the difference of the two axis,) LM be taken equal to CP , the point M will be in the periphery of the ellipse. Required a demonstration?

FIRST SOLUTION, by the Proposer.

This question will be best demonstrated by premising the following propositions.

PROPOSITION

PROPOSITION I. THEOREM, Fig. 132, 133, pl. 7.

Having described a circle on CH, the sum or difference of the axis of an ellipse as in the question, and having drawn CF to the extremity of any chord passing through B; then CP, the semi-diameter of the ellipse, will be equal to BG.

Join CG, CH; draw BR parallel to CG, BS parallel to GH, and PN perpendicular to AB. Then, on account of similar triangles,

$$CB^2 : BH^2 :: CS^2 : BR^2.$$

But BH = CD, and from the nature of the ellipse

$$CB^2 : CD^2 \text{ or } BH^2 :: AN \times NB : NP^2.$$

$$\text{Therefore } CS^2 : BR^2 :: AN \times NB : NP^2$$

and, by alternation $CS^2 : AN \times NB :: BR^2 : NP^2$.

Because the angle CGH is a right angle, therefore BR, parallel to CG, is perpendicular to GH. And because the angles PCN, BGR, in the same segment are equal to one another, therefore the right angled triangles PCN and BGR are similar :

$$\text{Therefore } GR^2 : CN^2 :: BR^2 : PN^2.$$

Consequently, by what was before proved,

$$CS^2 : AN \times NB :: GR^2 \text{ or } BS^2 : CN^2$$

and, by alternation, $CS^2 : BS^2 :: AN \times NB : CN^2$

$$\text{by composition, } BC^2 : BS^2 :: BC^2 : CN^2$$

Therefore BS = GR = CN. Therefore the two triangles CPN, BGR, are equal in all respects, and PC = BG.

Q. E. D.

COROL. If, in the line CF, CP be taken equal to BG, the point P will be in the circumference of the ellipse.

PROPOSITION II. LEMMA, Fig. 134, Pl. 7.

Let the two parts of the line AB, be respectively equal to the two parts of the line EG, viz. AC = EF and BC = FG; and further, let the angle ADC be equal to EHF, and the angle CDB to FHG : then will CD be equal to FH.

Upon AC, EF, let two segments of circles be described, each to contain an angle equal to the two equal angles ADC, EHF; and, in like manner, upon CB and FG, let segments be described, each to contain an angle equal to the two equal angles CDB and FHG. Then it is obvious that the circumferences of the two segments upon AC, CB will intersect one another

another in **C** and **D**; and the two circumferences of the segments on **EF**, **FG** will intersect one another in **F** and **H**. Now let the figure upon **EF** be applied to the figure on **AB**, so that **E** may be on **A**, and the line **EG** on the line **AB**; then will **F** be on **C**, and **G** on **B**; and the circumference of the segment on **EF** will coincide with the circumference of the segment on **AB** (Eu. 24, 111). And in like manner the circumference of the segment on **CD** will coincide with the circumference of the segment on **FG**: therefore since **F** coincides with **C**, **H** necessarily coincides with **D** (Eu. 10, 111). Therefore **FH** coincides with **CD**, and is equal to it.

Q. E. D.

DEMONSTRATION OF THE QUESTION. Fig. 132, 133, Pl. 7.

By Prop. 1, **GB** is equal to **CP**, and **BF** to **CQ**, therefore **FG** is equal to the sum of **CP** and **CQ** in the one case, and to the difference of **CP** and **CQ** in the other case.

Join **CM**, meeting the circumference again in **F**; draw the chord **FBU** and join **FU**, **GU**. Then it is easy to demonstrate (21 and 22, Eu. 111.) that the angle **FUB** is equal to the angle **MCK**, and the angle **BUG** to **MCL**: it is also clear that **LM** = **CP** = **BG**, and **MK** = **BF**, therefore, by the Lemma **CM** = **BU**. Therefore (Cor. Prop. 1.) the point **M** is in the circumference of the ellipse.

SCHOLIUM.

Prop. 1st. furnishes a good method of describing an ellipse by finding an indefinite number of points in the periphery. For any chords as **FG** and **TU**, being drawn through **B**, if **CF** and **CT** be drawn, and **CP**, **Cp** be taken each equal to **BG**, and **CM**, **Cm** each equal to **BU**: then **P**, **p**, **M**, and **m** will be points in the periphery of the ellipse.

SECOND SOLUTION, by Mr. LOWRY.

Fig. 135, 136, pl. 7. From **Q**, to the transverse axis **AB**, apply **QR** equal to the semi-conjugate axis **CD**, and produce it to meet **DE** in **I**; then it is proved by most writers on conic sections, (and is, in fact, the property on which the elliptical compasses are constructed, see Robertson's Conics, Prop. 24, B 11.)

11.) that IR is equal to the sum or difference of the semi-axis CB, CD, that is, equal to CH. Join GH, FH, and FC; then the angles CFH, ICR, being right angles, and the angles HFB, HCQ equal (Eu. 21 and 22. 111.), and $CH = IR$ and $CB = IQ$, the triangles BCF, QCI will be equal in every respect as may be very easily proved; and, consequently, BF is equal to CQ. It may be proved in a like manner that BG is equal to CP, and therefore FG is equal to the sum or difference of CP and CQ.

Again, let a circle be described through the points C, L, K, cutting CB in O, and CD in T, and join TO, OM, OK, and LK. Then because FG and LK are equal, and subtend the same angle at C, the circles CGFH, CKL will be equal; and the angles FCH, OTK, on the same arc OK, are equal. And the angles OTL, HCG, on the arc OL are equal, therefore the angles OKM, HFB, are equal (Eu. 21 and 22, 111.); and the triangles OKM, HFB, having the side $OK = HF$, the side $KM = FB$ or CQ , and the angle $OKM = HFB$, will have $OM = HB = CD$, and the angle $KOM = FHB$. Also the angle TOK is equal to the angle CHF, each being the complement of the angle OCF, and therefore the angles KOM, TOK, together are equal to two right angles, and the points T, O, M, are in a straight line. But since TCO is a right angle, TO is the diameter of the circle CKL, and equal to CH; and, consequently, TM and OM are respectively equal to the axis CB, CD, and therefore the point M is in the ellipse by the proposition referred to above.

It is evident from what has been already done, that the first part of the proposition is true when AB, DE, are any two diameters whatever, if instead of a circle being described on CH, as a diameter, a segment CGFH be described to contain an angle CFH equal to the angle ECB included between the diameters.

It is also obvious, from hence, how the axis may be found when two diameters of an ellipse are given. For if LK be interposed between the diameters CP, CQ, and equal to their sum or difference, and KM be taken equal to CQ, and a circle be described through the points C, L, K; then if MT be drawn through the centre of the circle meeting the circumference in O and T, MO and MT will be equal to the semi-axis, and their position will be determined by drawing CT and CO, and taking $CB = MT$, and $CD = MO$.

Ingenious Solutions were also received from Messrs. Bazley, Cavill, Simpson, and W. Smith.

XXIX. QUESTION 89, by PHILOMATHES OXONIENSIS.

Required the fluent of $\frac{x^3/x}{(1+\sqrt{x})^3}$, generated while x increafes from 0 to 2; and alfo the fluent of $\frac{x^3}{(x^2-1)^{3/2}}$, generated while x increafes from $\frac{1}{4}$ to 2?

FIRST SOLUTION, by J. H.

This queftion contains two problems, the firft of which may be folved thus:

Put $v^6 = x$; then $\sqrt{x} = v^3$, $\sqrt[3]{x} = v^2$, and $\dot{x} = 6vv^5$, and the given expreffion becomes $\frac{6vv^7}{(1+v^3)^3}$, the fluent of which is obvioufly $= \frac{6v^8}{8(1+v^3)^3} + \frac{6 \cdot 9 v^{11}}{8 \cdot 11(1+v^3)^4} + \frac{6 \cdot 9 \cdot 12 v^{14}}{8 \cdot 11 \cdot 14(1+v^3)^5} +$

$\frac{6 \cdot 9 \cdot 12 \cdot 15 v^{17}}{8 \cdot 11 \cdot 14 \cdot 17(1+v^3)^6}$, &c. *ad infinitum*, which feries will vanifh with v , and therefore needs no correftion, and, when $v^6 = 2$,

it will converge by the powers of $\frac{\sqrt{2}}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}}$ nearly, fo that its value may be obtained by computing a moderate number of terms. It may, however, be transformed into other feries which fhall converge much more fwifly. (See the *Philofophical Tranfactions* for the year 1798, Part I. Art. IX.) Or, the fluent of the above expreffion may be computed by a defcending feries

which fhall converge by the powers of $\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}}$ nearly.

But, fince the fluent is eafily attainable by logaram this and circular arches, which are feries already computed, I fhall now treat of that method.

It is eafy to perceive that the fluent of $\frac{6vv^7}{(1+v^3)^3}$ may be taken

in this form, viz. $\frac{Av^8}{(1+v^3)} + \frac{Bvv}{1+v^3} + \int \frac{Cvv}{1+v^3}$ in which
the

the coefficients A, B, and C, may be computed with great facility. But a better form for numerical calculation may be derived from Madam *Agnesi's* Theorem in her *Analytical Institutions*, Book III. Sect. I. Art. 56; observing, first, that the indices of x , in the numerator of the fraction which she assumes, will always be an arithmetical progression decreasing by the constant difference m : And, secondly, that when n (still using her notation) is greater than m , let m be taken from n as often as it may, which call p times, and put the remainder, $n - pm = r$; then will

$$\int \frac{\dot{x}x^n}{(x^m \pm a^m)^u} = \frac{Bx^{r+um-2m+1} + Cx^{r+um-3m+1} + Dx^{r+um-4m+1} + \&c.}{(x^m \pm a^m)^{u-1}}$$

+ $\int \frac{A\dot{x}x^r}{x^m \pm a^m}$, from the fluxion of which equation the values of A, B, C, D, &c. will easily be found, and then no more remains to be done than to compute the fluent of the term

$\frac{A\dot{x}x^r}{x^m \pm a^m}$ by the well known method of logarithms and circular arches.

Now, by means of this equation we shall find

$$\int \frac{6\dot{v}v^7}{(1+v^3)^3} \text{ to be } = -\frac{8v^5 + 5v^2}{3(1+v^3)^2} + \int \frac{10\dot{v}v}{3(1+v^3)}; \text{ and}$$

since the correct fluent of

$$\frac{\dot{v}v}{1+v} = \frac{1}{3} \text{ h.l. } \frac{\sqrt{(1-v+vv)}}{1+v} + \frac{1}{3} \times \left\{ \begin{array}{l} \text{cir. arch} \\ + \text{cir. a.} \end{array} \right\} \text{ to r. } \frac{\sqrt{3}}{2} \& \left\{ \begin{array}{l} \text{tang. } v-\frac{1}{2} \\ \text{tang. } +\frac{1}{2} \end{array} \right\},$$

we have the correct

$$\begin{aligned} \int \frac{6\dot{v}v^7}{(1+v^3)^3} &= -\frac{8v^5 + 5v^2}{3(1+v^3)^2} + \frac{10}{9} \text{ h. l. } \frac{\sqrt{(1-v+vv)}}{1+v} \\ &\quad + \frac{20}{9} \times \left\{ \begin{array}{l} \text{circ. arch} \\ + \text{circ. a.} \end{array} \right\} \text{ to rad. } \frac{\sqrt{3}}{2} \& \left\{ \begin{array}{l} \text{tang. } v-\frac{1}{2} \\ \text{tang. } +\frac{1}{2} \end{array} \right\}. \end{aligned}$$

And when $v^3 = 2$, these terms, expressed in numbers (according to my calculation) are as below.

$$\begin{aligned} -\frac{8v^5 + 5v^2}{3(1+v^3)^2} &= -1.175502 \\ \frac{10}{9} \text{ h.l. } \frac{\sqrt{(1-v+vv)}}{1+v} &= -0.764642 \\ &\quad -1.940144 \\ \frac{20}{9} \times \left\{ \begin{array}{l} \text{the two} \\ \text{circ. arches} \end{array} \right\} &= +2.207023 \\ &\quad +0.266879 = \text{the value required.} \end{aligned}$$

A solution of the second problem contained in this question, in algebraic terms, is attainable by Art. 61 of the 1st Sect. of the book above referred to, thus:

Put $x^3 - 1 = zx^3$; then, $x^3 = \frac{1}{1-z}$, $z = \frac{1}{(1-z)^{\frac{2}{3}}}$

$$z = \frac{\dot{z}}{3(1-z)^{\frac{2}{3}}}, \quad x^3 - 1 = \frac{z}{1-z}, \quad \text{and } (x^3 - 1)^{\frac{2}{3}} = \frac{z^{\frac{2}{3}}}{(1-z)^{\frac{2}{3}}}$$

by substitution, $\frac{\dot{x}x^3}{(x^3 - 1)^{\frac{2}{3}}} = \frac{\dot{z}}{3(1-z)^{\frac{2}{3}}} \times \frac{1}{1-z} \times \frac{(1-z)^{\frac{2}{3}}}{z^{\frac{2}{3}}} =$

$\frac{\dot{z}}{3z^{\frac{2}{3}}}$, the fluent of which is $-\frac{1}{4z^{\frac{4}{3}}}$; and since $z = \frac{x^3 - 1}{x^3}$, the

fluent is $-\frac{1}{4} \left(\frac{x^3}{x^3 - 1} \right)^{\frac{4}{3}} = -\frac{x^4}{4(x^3 - 1)^{\frac{4}{3}}}$. But, when $x =$

$\frac{11}{10}$, the fluent ought to be $= 0$; therefore the correct fluent is

$+ \frac{1}{4} \left(\frac{1331}{331} \right)^{\frac{4}{3}} - \frac{x^4}{4(x^3 - 1)^{\frac{4}{3}}}$; which, when x becomes $= a$,

will be $= \frac{1}{4} \left(\frac{1331}{331} \right)^{\frac{4}{3}} - \frac{4}{(7)^{\frac{4}{3}}} = 1.299882$ the value required.

SECOND SOLUTION, by LIMENUS, *Bruton*.

Beginning with the first expression, put $v^6 = x$, and then

$$\frac{\dot{x}^{\frac{3}{2}}/x}{(1 \pm \sqrt{x})^3} = \frac{6v^7 \dot{v}}{(1 + v^3)^3}. \quad \text{Assume}$$

$$A \times \text{fluent of } \frac{v\dot{v}}{1+v^3} - B \frac{v^2}{1+v^3} - C \frac{v^5}{(1+v^3)^2} = \text{fluent of } \frac{6v^7 \dot{v}}{(1+v^3)^3};$$

taking the fluxions and dividing by $\dot{v}$,

$$A \frac{v}{1+v^3} + B \frac{v^4 - 2v}{(1+v^3)^2} + C \frac{v^7 - 5v^4}{(1+v^3)^3} = \frac{6v^7}{(1+v^3)^3};$$

multiplying by $(1+v^3)^3$,

$$Av(1+v^3)^3 + B(v^4 - 2v)(1+v^3)^2 + C(v^7 - 5v^4) = 6v^7;$$

expanding the expression and collecting the homologous terms,

$$v^3(A+B+C-6) + v^4(2A-B-5C) + v(A-2B)$$

$= 0$, and by making the coefficients of the homologous terms $= 0$, we get

$$A =$$

$$A=2B; C=\frac{2A-B}{5}=\frac{3B}{5}; A+B+C=3B+\frac{3B}{5}=\frac{18B}{5}=6;$$

$$\text{whence } B=\frac{5}{3}; A=2B=\frac{10}{3}; \text{ and } C=\frac{3B}{5}=1: \text{ and there-}$$

$$\text{fore fluent of } \frac{6v^7\dot{v}}{(1+v^3)^3} = \text{fluent of } \frac{\frac{10}{3}v\dot{v}}{1+v^3} - \frac{\frac{4}{3}v^3}{1+v^3} - \frac{v^5}{(1+v^3)^3}.$$

Again assume

$$D \times \text{h. l. } (1-v+v^3) - F \times \text{h. l. } (1+v) + E \times \text{flu. } \frac{\dot{v}}{1-v+v^3} = \text{flu. } \frac{\dot{v}v}{1+v^3};$$

taking the fluxions and dividing by $\dot{v}$, we get

$$\frac{2Dv + E - D}{1-v+v^3} - \frac{F}{1+v} = \frac{v}{1+v^3}; \text{ multiplying by } 1+v^3,$$

$$v^3 \times (2D - F) + v \times (D + E + F) + E - D - F = v, \\ \text{or } v^3 \times (2D - F) + v \times (D + E + F - 1) + E - D - F = 0; \text{ and making} \\ \text{the coefficients of the homologous terms} = 0, \text{ there will be} \\ \text{had } 2D = F; E = D + F = 3D; \text{ and } D + E + F = 6D \\ = 1; \text{ whence } D = \frac{1}{6}, E = 3D = \frac{1}{2}, F = 2D = \frac{1}{3}, \text{ and} \\ \text{therefore}$$

$$\text{fluent of } \frac{v\dot{v}}{1+v^3} = \frac{1}{6} \times \text{h. l. } (1-v+v^3) - \frac{1}{3} \times \text{h. l. } (1+v) + \text{flu. } \frac{\dot{v}}{1-v+v^3}$$

$$= \frac{1}{6} \times \text{h. l. } \sqrt{\frac{1-v+v^3}{(1+v)^3}} + \text{flu. } \frac{\dot{v}}{1-v+v^3}$$

$$= \frac{1}{6} \times \text{h. l. } \sqrt{\frac{1+v^3}{(1+v)^3}} + \frac{R\sqrt{3}}{3} = \frac{1}{6} \times \text{h. l. } \frac{1+v^3}{(1+v)^3} + \frac{R\sqrt{3}}{3},$$

where R denotes the length of a circular arch radius 1, and tangent $\frac{v\sqrt{3}}{2-v}$; and, consequently,

$$\text{fluent of } \frac{6v^7\dot{v}}{(1+v^3)^3} = \frac{10}{3} \times \text{flu. } \frac{\dot{v}v}{1+v^3} - \frac{\frac{4}{3}v^3}{1+v^3} - \frac{v^5}{(1+v^3)^3} \\ = \frac{4}{3} \times \text{h. l. } \frac{1+v^3}{(1+v)^3} + \frac{10R\sqrt{3}}{9} - \frac{\frac{4}{3}v^3}{1+v^3} - \frac{v^5}{(1+v^3)^3} \\ = .266866 \text{ in the present case where } x=2, \text{ or} \\ v = x^{\frac{1}{3}} = 2^{\frac{1}{3}} = 1.122462.$$

Proceeding now to the second expression, put $\frac{x^3}{x^3-1} = v^3,$

$$\text{and taking the fluxions } \frac{-3x^2\dot{x}}{(x^3-1)^2} = 3v^2\dot{v}, \text{ or } \frac{x^2\dot{x}}{(x^3-1)^2} = -v^2\dot{v};$$

(Z 2)

also

also $\frac{x}{(x^3 - 1)^{\frac{1}{3}}} = v$; multiplying these two expressions together,

$$\frac{x^{\frac{2}{3}}}{(x^3 - 1)^{\frac{1}{3}}} = -v^3 \dot{v}; \text{ therefore fluent of } \frac{x^{\frac{2}{3}}}{(x^3 - 1)^{\frac{1}{3}}} = -\frac{v^4}{4}.$$

$$-\frac{v^4}{4} = 1.5986011, \text{ when } x = \frac{11}{10}$$

$$-\frac{v^4}{4} = 0.2987187, \text{ when } x = 2,$$

therefore 1.2998824, or 1.3 nearly the difference of the preceding values of $-\frac{v^4}{4}$ is the fluent of $\frac{x^{\frac{2}{3}}}{(x^3 - 1)^{\frac{1}{3}}}$, generated

whilst x from $\frac{11}{10}$ increases to 2.

THIRD SOLUTION, by Mr. GRIFFITH WILLIAMS.

1. To find the fluent of $\frac{\dot{x}^{\frac{2}{3}} \sqrt{x}}{(1 + \sqrt{x})^3}$.

Assume $x = v^6$, then $\dot{x}^{\frac{2}{3}} = v^2$, $\sqrt{x} = v^3$, and $\frac{\dot{x}^{\frac{2}{3}} \sqrt{x}}{(1 + \sqrt{x})^3} = \frac{6v^7 \dot{v}}{(1 + v^3)^3}$.

By Form. 7, Case 2, of Emerson's Fluxions, the fluent of

$$\frac{v \dot{v}}{1 + v^3} \text{ is } = \frac{\sqrt{3}}{3} \times \text{circ. arc. rad. 1 and sine } \frac{\frac{1}{2} \sqrt{3} v}{\sqrt{(1 - v + v^3)}}$$

$$= \frac{1}{3} \times \text{h. l. } \sqrt{(1 - v + v^3)} - \frac{1}{3} \times \text{h. l. } (1 + v): \text{ put this}$$

fluent = A; then by forms 12 and 14 of the same book, the

$$\text{fluent of } \frac{6v^7 \dot{v}}{(1 + v^3)^3} \text{ is } = \frac{10}{3} A - \frac{5}{3} \times \frac{v^3}{1 + v^3} - \frac{v^5}{(1 + v^3)^2},$$

which expression vanishes when $x = 0$, and when $x = 2$, the required fluent is = .266992.

2. To find the fluent of $\frac{x^{\frac{2}{3}}}{(x^3 - 1)^{\frac{1}{3}}}$.

Assume $x^3 = \frac{v^3}{v^3 - 1}$; then $\frac{x^{\frac{2}{3}}}{(x^3 - 1)^{\frac{1}{3}}} = -v^3 \dot{v}$, the fluent of

which

which is $-\frac{v^4}{4}$; hence the correct fluent in this case is 1.9 nearly.

XXX. PRIZE QUESTION 90, by EDINBURGENSIS.

Suppose a horse to move with a given velocity in the circumference of a given circle. Required the angle which the body of the horse will make with the horizon by leaning towards the centre ?

FIRST SOLUTION, by Mr. THOMAS BAZLEY.

This question appears to be equivalent to the following one. AGB (fig. 137, pl. 7.) is a given circle, and its centre is C; CE an axis perpendicular to the plane of the circle at C: AP is a right line loaded with a weight in the given point P: this right line is moveable upon the point A in a vertical plane passing through CE. Now when the right line AP is revolved in the circle AGB with a given velocity, it is required to determine the angle which AP makes with the plane of the circle, so that it may remain in equilibrio with respect to CB.

Draw the diameter ACB and EPF parallel thereto, drop the perpendicular PD on CA, and draw AF parallel to PD to meet EPF in F. Then it is plain from the composition of forces, that if PF is to PD, as the centrifugal force at P, to the force of gravity; then, PA has the required position.

This being premised, put $CA = r$, $AP = a$, $CD = x$, and $g = 16\frac{1}{2}$ feet, also $v =$ velocity of the point A. The velocity of A is to the velocity of P :: CA : CD or EP; hence,

the velocity of P $= \frac{vx}{r}$. By the laws of central forces, we have, as the force of gravity is to the centrifugal force at P so is $2g : \frac{v^2x}{r^2} :: PD : PF$ or DA :: $\sqrt{a^2 - (r-x)^2} : r-x$;

hence we obtain

$$4g^2(r-x)^2 = \frac{v^4}{r^4} x^2 \{a^2 - (r-x)^2\}$$

and by reduction

$$x^4 - 2rx^3 + \left(\frac{4r^4}{v^4}g^2 - a^2 + r^2\right)x^2 - \frac{8r^5}{v^4}g^2x + 4g^2\frac{r^6}{v^4} = 0.$$

From

From this equation, x being determined, DP and DA become known; and, consequently, the angle DAP which was required.

SECOND SOLUTION, by PHILOMATHES OXONIENSIS.

If the radius of the circle described by the centre of gravity of the horse's body be denoted by a , the length of the arch of the circle described by the aforesaid centre of gravity in one second, by b , and the force of gravity by f ; then, by mechanics *, the centrifugal force of the aforesaid centre of gravity will be to the force of gravity as $\frac{bb}{a}$ to f . Now, if the sine of the angle of the horse's inclination to the horizon be called x , we shall have, by the resolution of forces, and the condition of the question, this equation, viz. $\frac{bbx}{a} = f \sqrt{1 - xx}$; and thence, $\frac{x}{\sqrt{1 - xx}} = \frac{af}{bb}$, the tangent of the angle required.

Example. If the radius of the circle be 15 feet, and the horse's speed at the rate of 5 miles per hour, b will be $= \frac{22}{3}$; and since f is $= 32 \frac{1}{6}$, the tangent of the horse's inclination to the horizon will be $= \frac{9 \times 15 \times 32 \frac{1}{6}}{22 \times 22} = 8.97211$, answering to $89^\circ 38' \frac{1}{2}$ nearly.

According to one or other of these methods were the Solutions given by Messrs. Codling, Collins, Jones, Myers, Pickering, Roville, Senex, and Wilton.

SENEX is requested to send to Mr. GLENDINNING's for the Medal for solving the Prize Question.

* See the Ladies' Diary for 1783, p. 39.

NOTICES

RELATING TO MATHEMATICS.

I. DEATH OF DR. ROBISON.

IT is with regret that we announce the death of that justly celebrated mathematician and philosopher, Dr. John Robison, Professor of Natural Philosophy in the University of Edinburgh. This event took place on the 30th of January last *. A few months before his death he gave to the world the first volume of a work, entitled, *ELEMENTS OF MECHANICAL PHILOSOPHY*, being the substance of a Course of Lectures which he had delivered annually in the discharge of his academical duty, ever since the year 1774. Such as have had the pleasure of hearing these Lectures, or have read the numerous and valuable articles which he has contributed to the last edition of the *Encyclopædia Britannica*, will naturally suppose that this work must possess no common degree of excellence, and that it will far transcend most of the treatises that have appeared under the name of *Elements of Natural Philosophy* for many years past in this country. We venture to assure them, that upon perusing it they will not find their expectations disappointed. A second volume of the work we understand was in the press when the Author died; and it will be published in the course of the present summer.

Dr. Robison has been succeeded in the Professorship of Natural Philosophy by Mr. John Playfair, Professor of Mathematics in the same University, with whose merits as a Mathematician and Philosopher, as well as an elegant and correct writer, we cannot suppose any of our readers to be unacquainted.

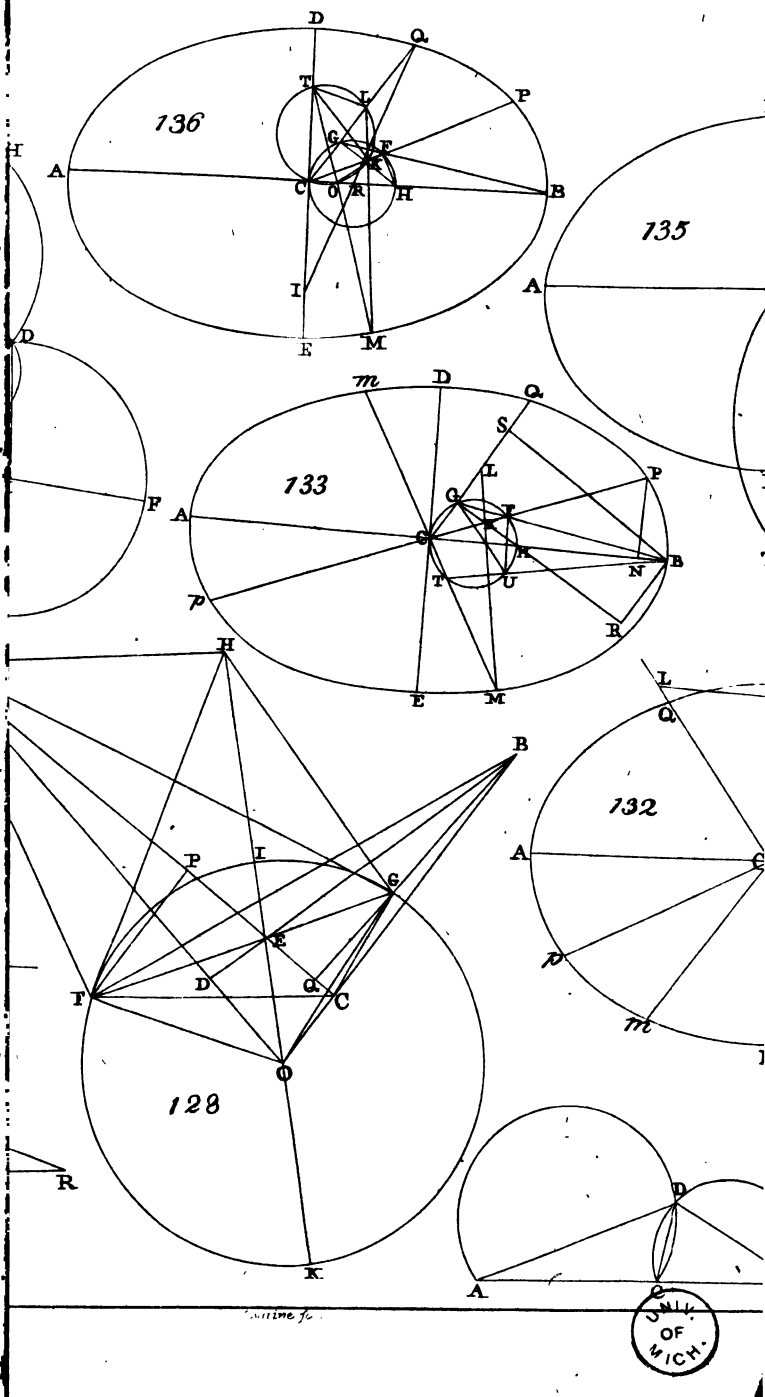
Mr. Playfair has been succeeded in the Mathematical Chair by John Leslie, Esq. a gentleman of great Mathematical eminence, and Author of an *Experimental Inquiry into the Nature and Propagation of Heat*, a work which has justly acquired him great celebrity as an Experimental Philosopher, and for which the Royal Society of London has awarded to him the biennial gold medal placed at their disposal by the munificence of Count Rumford.

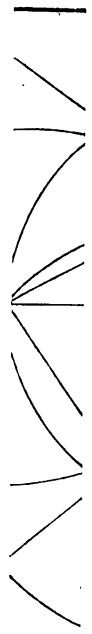
* We suppose his age to have been about 65.

II. MATHEMATICAL WORKS IN THE PRESS.

IT has often been remarked as a singular circumstance, that during the last half century, while the practice of mechanics and the structure and operation of machines have received so many and such valuable improvements in this country, we have only had one treatise (that by Emerfon) into which we can look for information both on the theory and the actual construction of machinery. Mr. Gregory of the Royal Military Academy, Woolwich, has endeavoured to supply the deficiency just adverted to, and has now in the press a general Treatise of Mechanics, which is intended to be comprized in two volumes, octavo. The first volume will be devoted chiefly to the theory, and will be divided into five books, under the several heads of Statics, Dynamics, Hydrostatics, Hydrodynamics, and Pneumatics. The second volume will be chiefly appropriated to the description of machinery, and will commence with some practical remarks on the application, improvement, and simplification of mechanical contrivances; on friction, the stiffness of ropes, the energy of different first movers, &c. And these will be followed by accounts, arranged alphabetically, of about one hundred of the most curious, useful, and important machines. In this latter part Mr. Gregory has been promised the assistance of some celebrated civil engineers; and the alphabetical arrangement (the only unfinished part of the work,) will be completed in the course of the month of July, when he hopes he shall have viewed the communications of these gentlemen, or of any others who may favour him with descriptions of new and useful machines. The work is intended to be published before the end of the present year.







ARTICLE IV.

Solutions to Questions proposed in Number III.

I. QUESTION 91, by Capt. GEO. GORRY.

Passing along a straight and level road, near a very lofty tower, and on the same horizontal plane with its base, I wished to ascertain its height; but being prevented by fences from quitting the road, and having no instrument but one for taking vertical angles, I proceeded thus: at a convenient point on the road, I found the angle of elevation of the top of the tower, $30^{\circ} 48'$; then measuring 60 yards along the road, I found a second angle of elevation $40^{\circ} 33'$; at the end of another 60 yards in the same direction, I intended taking a third angle of elevation, but the tower was hidden by a high wall; I therefore measured on to 72 yards from the second station, and there found the angle of elevation of the top of the tower, $50^{\circ} 23'$. From these data, the height of the tower, and its nearest distance from each of the three stations are required?

FIRST SOLUTION, by Mr. WM. MELLOR, *London.*

Suppose AED (fig. 164. pl. 9.) the horizontal plane, EF the tower perpendicular thereto; let A, B, C be the three places of observation. Join AE, BE, CE; AF, BF, CF; and on AC demit the perpendicular ED. Then we have given $AB = 60$ and $BC = 72$ yards; the angle $FCE = 50^{\circ} 23'$, the angle $FBE = 40^{\circ} 33'$, and the angle $FAE = 30^{\circ} 48'$, to find the distances AE, BE, CE; and EF the height of the tower.

Since the triangles AEF, BEF, CEF are all right angled at E, it is obvious that the distances AE, BE, CE will be as the co-tangents of the given angles FAE, FBE, FCE: let these co-tangents be represented by a, b, c respectively; then we have to determine the point E such that the lines AE, BE, CE may have the same ratio as a, b, c .

This may be effected a variety of ways, but as simple a method as any is that in Simpson's Select Exercises, page 130; which is nearly as follows.

Take BM equal to BC, BN to AB, and make

$$MG : BM (BC) :: a : b, \text{ and}$$

$$BN (AB) : NG :: b : c.$$

With the lines MN, MG, NG form the triangle MNG, and join BG. Draw AE to make the angle EAB = MGB, meeting BG produced in E the point required.

For by the similar triangles AEB, GMB,

$$AE : BE :: MG : MB :: a : b; \text{ and}$$

$$BE : BA (BN) :: MB : BG.$$

Therefore the triangles BEC, BGN are similar, and therefore $BE : EC :: BN : NG :: b : c$; wherefore it is manifest that the distances AE, BE, CE are respectively as a, b, c .

The calculation follows very easily from the construction; for in the triangle MGN all the sides are given to find the angle GMN, which is equal to the angle AEB; then in the triangle MGB two sides and the included angle are given to find the angle MGB, which is equal to the angle EAB. Hence in the triangle AEB we find $AE = 159.087$, $BE = 110.8414$; and then $CE = 78.507$ yards. Finally, in the right-angled triangle AEF we have the side AE and the angle FAE, to find EF, the height of the tower, equal to 94.835 yards.

SECOND SOLUTION, by Mr. BAZLEY, Bolton.

Let $AB = 60 = r$, $BC = 73 = s$; also let the cotangents of the angles FAE ($30^\circ 48'$), FBE ($40^\circ 33'$), FCE ($50^\circ 23'$), be represented by the letters a, b, c respectively.

Then, by trigonometry, putting $EF = x$, we have,
 $1 : a :: x : ax = AE$, $1 : b :: x : bx = BE$, $1 : c :: x : cx = CE$;
 and, Euc. 12. 11. $a^2 x^2 = b^2 x^2 + r^2 + 2r \times BD$; hence

$$BD = \frac{a^2 x^2 - b^2 x^2 - r^2}{2r}. \text{ In like manner}$$

$$CD = \frac{b^2 x^2 - c^2 x^2 - s^2}{2s} = BD - s; \text{ whence } BD = \frac{b^2 x^2 - c^2 x^2 + s^2}{2s}.$$

$$\text{Therefore } \frac{b^2 x^2 - c^2 x^2 + s^2}{2s} = \frac{a^2 x^2 - b^2 x^2 - r^2}{2r}. \text{ Hence}$$

$$x^2 = \frac{rs^2 + sx^2}{s(a^2 - b^2) - r(b^2 - c^2)}, \text{ and } x = \sqrt{\frac{rs(r+s)}{s(a^2 - b^2) - r(b^2 - c^2)}} = 94.835$$

yards. Also $AE = 159.087$, $BE = 110.8414$, and $CE = 78.507$ yards.

The Proposer also answered it.

II. QUESTION 92, *by* NEGOTIATOR.

A sells goods to B, on condition to allow 10 per cent. discount, if he paid in six months.—What discount ought A to allow, if payment be made him in two months; interest of money being 5 per cent. per annum, simple interest?

FIRST SOLUTION, *by* Mr. BAZLEY.

If s be put to denote the value of the goods, then the discounted sum paid at the end of six months is $\frac{9}{10} s$. Let x be the sum which B ought to pay if the goods be paid for at the end of two months; then it is plain that x at simple interest for the remaining four months should amount to $\frac{9}{10} s$: that is (by the rate of interest given by the question,

$$x + \frac{x}{60} = \frac{9}{10} s.$$

From this equation we have $x = \frac{54s}{61}$; and, consequently,

$s - x = \frac{7s}{61}$, or £11.4754 per cent. the discount required.

SECOND SOLUTION, *by* A. M.

Since B is to pay only 90 per cent. for his goods if he pays A in six months, therefore the sum per cent. which B ought to pay if payment be made to A in two months, is such a sum as being put out at 5 per cent. simple interest for four months will amount to 90 £. Hence $90 \div (.3 \times .05) + 1 = 88.524$ £. = 88 £. 10s. $5\frac{1}{4}$ d. per cent. B ought to give A for his goods if payment be made in two months, or 100 £.—88 £. 10s. $3\frac{1}{4}$ d. = 11 £. 9s. $6\frac{1}{4}$ d. per cent. is the discount required.

Mr. Mellor answered this question.

III. QUESTION 93, *by* Mr. WM. SIMPSON, Bolton.

If a circle be described through the two foci of an ellipse, and any point in the conjugate axis produced; then a right line being drawn from that point, to one of the points where the circle cuts the ellipse, will be a tangent at that point. Required the demonstration?

AAa

SOLUTION, by Mr. J. JOHNSON, Birmingham.

Fig. 165. pl. 9. Let a circle which passes through the foci f, F , meet the conjugate axis in E and H and the ellipsis in P , and draw fP, FP, HP , and let EP produced meet the transverse axis in G . Then it is evident that the arc fHF is bisected in H , therefore the angle FPH is equal to the angle SPH ; and the angles HPG, HPE are, each of them, right angles. If the former angles be taken from the latter, the remaining angle FPG will be equal to the remaining angle fPE , therefore EP is a tangent to the ellipse as is proved by all the writers on conic sections.

Several properties of the ellipse may be immediately deduced from the known properties of the triangle fPF and its circumscribing circle $FPEf$. We select a few of the most obvious.

1. $fP : FP :: fK : FK$.
2. $fP \cdot PF = KP \cdot PH = IC \cdot HE$.
3. Let HL be drawn perpendicular to fP . Then $LP =$ the semi-transverse axis $=$ half the sum of fP, PF ; and $FL =$ half the difference of fP, PF .
4. If LC be drawn to the centre, it will be parallel to the tangent at P .
5. Let PI be parallel and PO perpendicular to AB .
Then $fP + PF$ or $AB : fF :: 2CO : fP - PF$;
or $AB : fF :: CO : fL$.
6. $AC^2 = LP^2 = HI \cdot EC$.
7. $(fP - PF)^2 = CH \cdot IE$.
8. $AC^2 : CD^2 :: HP : PK$.

Messrs. Bazley, Mellor, and A. M. likewise sent solutions to this question.

IV. QUESTION 94. by CENTROBARICOS.

Suppose a cone, whose axis is 10, and diameter 8 inches, be made to vibrate, its vertex being the point of suspension; it is required to ascertain at what distance from its vertex, on the flant side, it must strike an immovable obstacle, so that all its angular motion shall be destroyed, or that its point of suspension shall not be affected by the stroke?

SOLUTION,

SOLUTION, by Mr. BAZLEY.

Let ABC (fig. 166. pl. 9.) be the cone whose vertex is C, and axis CD; then if O be the centre of percussion and OP perpendicular to the side CB, it is plain from the principles of mechanics (Emerfon's Mechanics 4, to prop. 9.) that P is the required point.

Now it is shewn by the writers on fluxions that the distance $CO = \frac{4CD^2 + DB^2}{5CD}$, and by the similar triangles COP, CDB, we have

$$CB : CD :: CO : CP :: \frac{4CD^2 + DB^2}{5CD} : \frac{4CD^2 + DB^2}{5CB}$$

In the present case $CD = 10$, $DB = 4$, $CB = \sqrt{(100 + 16)}$
 $= 10.77033$ and $CP = \frac{4CD^2 + DB^2}{5CB} = 7.725$ inches,
 the distance required.

V. QUESTION 95, by DIOPHANTUS.

It is required to find three rational numbers, x , y , and z , such that the formula

$x^2 + y^2 + 2z^2$, $x^2 + z^2 + 2y^2$, and $y^2 + z^2 + 2x^2$,
 may all be squares?

SOLUTION, by Mr. CUNLIFFE, R. M. College.

Put $z + m + n = x$, and $z + m = y$, then

$x^2 + y^2 + 2z^2 = 4z^2 + 2z(2m + n) + 2m^2 + 2mn + n^2 = \text{a square, and}$

$x^2 + z^2 + 2y^2 = 4z^2 + 2z(3m + n) + 3m^2 + 2mn + n^2 = \text{a square.}$

Assume $2z + r$ for the root of the former, and $2z + s$ for the root of the latter of these expressions, that is, put

$4z^2 + 2z(2m + n) + 2m^2 + 2mn + n^2 = (2z + r)^2 = 4z^2 + 4rz + r^2$, and

$4z^2 + 2z(3m + n) + 3m^2 + 2mn + n^2 = (2z + s)^2 = 4z^2 + 4sz + s^2$,

from whence

$$z = \frac{2m^2 + 2mn + n^2 - r^2}{2(2r - 2m - n)} = \frac{3m^2 + 2mn + n^2 - s^2}{2(2s - 3m - n)}$$

equating the numerators and denominators of these fractions to make them identical, we shall thence obtain

$$s^2 - r^2 = m^2, \text{ and } s - r = \frac{m}{2}$$

dividing

dividing the former of these by the latter, $s + r = 2m$; whence

$s = \frac{5m}{4}$; and $r = \frac{3m}{4}$, and by means of these

$$z = \frac{2m^2 + 2mn + n^2 - r^2}{2(2r - 2m - n)} = \frac{2m^2 + 2mn + n^2 - \frac{9m^2}{16}}{2(m + 2n)} = \frac{23m^2 + 32mn + 16n^2}{16(m + 2n)},$$

$$y = z + m = \frac{23m^2 + 32mn + 16n^2}{16(m + 2n)} + m = \frac{16m^2 + 16n^2}{16(m + 2n)}, \text{ and}$$

$$x = z + m + n = \frac{7m^2 - 16mn - 16n^2}{16(m + 2n)}.$$

Omitting the common denominator $16(m + 2n)$ we may take

$$z = 23m^2 + 32mn + 16n^2,$$

$$y = 7m^2 + 16n^2 \text{ and}$$

$$x = 7m^2 - 16mn - 16n^2;$$

these values make the two first of the proposed expressions squares. Therefore there only remains the last of the said expressions, viz. $y^2 + z^2 + 2x^2$, to be made a square.

By writing for x , y and z the values just deduced, we shall have, $y^2 + z^2 + 2x^2$

$$= (7m^2 + 16n^2)^2 + (23m^2 + 32mn + 16n^2)^2 + 2(7m^2 - 16mn - 16n^2)^2$$

$$= 676m^4 + 1024m^3n + 2048m^2n^2 + 2048mn^3 + 1024n^4$$

$$= 4 \times (169m^4 + 256m^3n + 512m^2n^2 + 512mn^3 + 256n^4)$$

= a square. Therefore

$$169m^4 + 256m^3n + 512m^2n^2 + 512mn^3 + 256n^4 = \text{a square.}$$

Assume $13m^2 + \frac{128}{13}mn + 16n^2$ for its root; that is, put

$$169m^4 + 256m^3n + 512m^2n^2 + 512mn^3 + 256n^4$$

$$= (13m^2 + \frac{128}{13}mn + 16n^2)^2$$

$$= 169m^4 + 256m^3n + \frac{(128)^2 m^2 n^2}{169} + 416m^2 n^2 + \frac{4096 mn^3}{13} + 256n^4$$

which after proper reduction gives $\frac{m}{n} = \frac{208}{1}$, take $m = 208$

then $n = 1$, and hence

$$x = 7m^2 - 16mn - 16n^2 = 16 \times 18719$$

$$y = 7m^2 + 16n^2 = 16 \times 18929$$

$$z = 23m^2 + 32mn + 16n^2 = 16 \times 62609.$$

And

And as these numbers are all divisible by 16, we may take
 $x = 18719$, $y = 18929$ and $z = 62609$.

VI. QUESTION 96, by Mr. JOHN DAWES, Birmingham.

In Mr. Watt's steam engine, the reciprocating motion of the engine, is converted into a circular motion by a peculiar contrivance, consisting of two equal wheels that work into one another: it is required to demonstrate that the central wheel will turn round twice for every revolution of the planet wheel about it, or for every complete stroke of the engine; and also to define what will be the motion of the central wheel when it is not equal to the planet wheel, but in any given proportion to it?

FIRST SOLUTION, by Mr. DAWES, the Proposer.

The ratio of the times of revolution of the central and planet-wheels, is equal to the ratio of the radius of the central wheel, and the sum of the radii of the central and planet wheels.

Let C, (fig. 167. pl. 9.) represent the central, and P, L, A, N, four positions of the planet-wheel; then, it is evident that the centre of the planet-wheel revolves in a circle whose radius is equal to the sum of the radii of the central and planet-wheels; and (Emerson's Algebra, page 450) the locus of any point (p) in the circumference of the planet wheel (which has no motion round its own axis) is a circle equal to that which is the locus of the centre of the said wheel; but as both wheels, at the point of contact, move with the same velocity, it follows that any two points, one in the periphery of each wheel, must move through the same space in equal times; therefore, in equal times, it will be inversely, as the revolutions of the central, is to the circumvolutions of the planet, so is the radius of the central, to the radius of the circle formed by either the centre or any point in the extremity of the planet-wheel.

Q. E. D.

Observation. The above demonstration is upon the supposition that the connecting rod which suspends the planet-wheel, always moves parallel to itself; but in reality the end is connected to the beam, which moves upon a fixed fulcrum, therefore it cannot move parallel to itself, but makes an angle at the end of the beam which destroys much of the power of the engine.

SECOND

SECOND SOLUTION, by Mr. BAZLEY.

Let CAF (fig. 168, pl. 9.) be the central-wheel, C the centre, AOB, a diameter of the revolving or planet-wheel, O, its centre which describes the circle OE, concentric to AF. Suppose the wheels fastened together at their point of contact A, and let the motion commence at the position CAB. Now if the whole thus fixed be conceived to describe round C the angle BCD, then it is plain that the diameter AB will have come to the position FD. But by the nature of the question the diameter AB is moved always parallel to itself; and therefore that FD may have the position ba, parallel to BA, it is evident that the arc aF of the planet-wheel must impel the central-wheel through an equal arc; so that the arc through which the central-wheel has passed, whilst the centre of the planet-wheel has described the arc OE, is $AF + Fa$: and it remains to find the angle answering to that arc of the central-wheel. Put $\angle AC = a$, $AB = b$, and the $\angle OEC = FEA = ACF = n$; then, by the nature of the circle, $a : b :: n : \frac{bn}{a}$ = the angle of the central whose arc is $= Fa$; hence, whilst the revolving wheel has passed through the $\angle n$, the central-wheel has described the $\angle \left(n + \frac{bn}{a} \right) = \frac{a+b}{a} \times n$; so that whilst the planet wheel makes one revolution, the central-wheel makes $\frac{a+b}{a}$ turns on its axis. If a and b , are equal, $\frac{a+b}{a} = 2$.

VII. QUESTION 97, from *Geometrie Descriptive*, by MONGE.

If four unequal spheres be any how situated in space, and every two of them be circumscribed by a cone, so that its vertex may be on the same side of the centres of the spheres; the vertices of the six cones thus constituted shall be at the intersections of four straight lines situated in a plane: and if every two of the spheres be circumscribed by another cone, having its vertex between the centres of the spheres; the vertices of these cones, taken three by three, shall be in the same plane as three of the former. Required the demonstration?

SOLUTION,

SOLUTION, by Mr. LOWRY, R. M. College.

Let A, B, C, D, (fig. 159, pl. 9.) be the centres of four unequal spheres any how situated in space; P, Q, R, S, T, U, the vertices of the cones circumscribing every two of them taken in the order A, B; A, C; A, D, &c. each vertex being on the same side of the centres of the spheres; and p, q, r, s, t, u , the vertices of the circumscribing cones, taken in the same order, but situated between the centres of the spheres.

Then, it is evident that each of these vertices is in the straight line joining the centres of the two spheres which are circumscribed by the respective cones. Let the lines PBA, RDA, TDB, be drawn from the vertices P, R, T, to the centres of the spheres A, B, D; and conceive a plane to pass through the points, A, B, D; then it is evident that the vertices, P, R, T, will also be in that plane. Join PR, TR, and draw BV parallel to AD meeting PR in V, and let the radii of the sphere whose centres are A, B, C, D, be denoted by a, b, c, d , respectively.

Then by the property of tangents,

$$a : b :: AP : BP :: AR : BV, \text{ and}$$

$$a : d :: AR : DR; \text{ therefore}$$

$$b : d :: BV : DR; \text{ but}$$

$$b : d :: BT : DT; \text{ therefore}$$

$BV : DR :: BT : DT$. From whence it is evident that the points P, V, R, T, are in a straight line.

Hence, it follows, that if three unequal spheres be any how situated in space, and every two of them be circumscribed by a cone, the vertices of the three cones thus constituted will be in the same straight line.

From this we may infer that the vertices P, Q, S, of the cones circumscribing the spheres A, B, C, are in the straight line PS; that the vertices U, Q, R, of the cones circumscribing the spheres A, C, D, are in the straight line UR; and that the vertices U, S, T, of the cones circumscribing the spheres B, C, D, are in the straight line UT. Therefore the vertices of the six circumscribing cones are at the intersections of the four straight lines PT, PS, UR, and UT.

And these straight lines are all in the same plane: for if a plane be conceived to pass through the points P, T, U, the points R, S, will necessarily be in that plane, and consequently the point Q will also be in that plane.

Again, if BV meet Pr in v, it will appear, exactly as in the first part of the demonstration, that the vertices P, t, r, are in the straight line Pr, and that the vertices T, r, p, are in the

straight line Tp ; therefore it is obvious that the vertices p, t, r , are in the same plane as the vertices P, T, R , or in the plane passing through the centres A, B, D . In the same manner it appears that the vertices p, q, s , are in the same plane as the vertices P, Q, S ; that the vertices u, q, s , are in the same plane as the vertices U, Q, S ; and, finally, that the vertices u, s, t , are in the same plane as the vertices U, S, T .

COROL. It follows immediately from the above, that "If the tangents drawn to every two of three unequal circles be produced till they meet, the points of intersection will be in a right line."

The next Ten Questions, taken from the Cambridge University Calendar, are selected from the Senate House Problems, proposed to Candidates for the Degree of B. A. in January, 1804.

VIII. QUESTION 98, from the CAMB. UN. CAL.

Three known rectilinear objects, placed contiguous, and in the same right line, appeared to the eye of a spectator of the same length; find his position?

FIRST SOLUTION, by Mr. MELLOR, London.

Let AB, BC, CD , (fig. 170, pl. 9.) represent the three rectilinear objects, placed contiguous and in the same straight line. And let E be the position of the spectator's eye when the objects appear of the same length. Draw AE, BE, CE, DE . Then by the principles of optics, the objects AB, BC, CD , will be apparently of the same length when the subtended angles AEB, BEC, CED , are equal.

Therefore, Euc. vi. 3. $AE : CE :: AB : BC$

and $BE : DE :: BC : CD$:

Produce AC to G , so that $CG : AG :: BC : AB$, and on BG as a diameter describe a circle B, E, G, E' , which is well known to be the locus of E , when AE has to CE the same ratio as AB to BC . Again produce BD to H , so that $DH : BH :: BC : CD$, and a circle described on the diameter CH will be the locus of the point E when BE has to ED the same ratio as BC to CD ; therefore the intersection of these circles will determine the point E .

If

If a circle be described on the line EE' , which joins the points of intersection of the circles, as a diameter, and having its plane perpendicular to the line AH , it is obvious that at any point whatever in the circumference of this circle the three objects will appear of the same length.

SECOND SOLUTION, by Mr. BAZLEY.

Put $AB = a$, $BC = b$, $CD = c$, $AE = v$, $BE = x$, $CE = y$, and $DE = z$.

Then $a : b :: v : y = \frac{bv}{a}$, and $b : c :: x : z = \frac{cx}{b}$;

Also (by Playfair's Euclid, prop. B, book vi.)

$vy = ab + x^2 = \frac{bv^2}{a}$; hence $v^2 = \frac{a^2b + ax^2}{b}$. Again

$zx = bc + y^2$, or $\frac{cx^2}{b} = bc + \frac{b^2v^2}{a^2} = bc + \frac{ab^3 + bx^2}{a}$; or

$cax^2 = ab^3c + ab^3 + b^2x^2$: whence $x = \sqrt{\frac{ab^3c + ab^3}{ca - b^2}}$;

from which it appears that if b^2 exceeds ca , that is, if BC^2 be greater than $AB \times CD$, the thing is impossible.

IX. QUESTION 99, from the CAMB. UN. CAL.

Shew how the circumference of an ellipse may be computed, of which the excentricity is .99, the semi-axis major being 1?

SOLUTION,

In consequence of the serieses commonly given, in our Treatises of Fluxions, &c. for finding the perimeter of an ellipse, converging so very slowly when the excentricity is great as to be entirely useless, Mathematicians have been induced to consider the subject at greater length, which has led to the discovery of several series that converge quickly for every degree of excentricity that can possible take place. As some of these series may probably be unknown to many of the readers of the Repository, it may not be improper to select such of them as are most remarkable for their elegance or practical utility.

FIRST SERIES, by Mr. Ivory, inserted in the 4th. Vol. of the Transactions of the Royal Society of Edinburgh.

Half the perimeter =

$$\frac{\pi}{1+d} \left\{ 1 + \frac{1^2}{2^2} \cdot d^2 + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} \cdot d^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot d^6 + \&c. \right\}^*$$

where 1 and c are the semi-axes, $d = \frac{1-c}{1+c}$, and $\tau = 3.14159$.

This series converges with great rapidity when the excentricity is small.

SECOND SERIES, given by Legendre in *Mem de l'Acad.* and by Euler, *Novi Comm. Petrop.* Tom. XVIII.

One fourth of the perimeter =

$$\sqrt{\left(1 - \frac{e^2}{2}\right)} \frac{\pi}{2} \left\{ 1 - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1}{2} e^2 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3}{2 \cdot 4} e^4 - \&c. \right\}$$

where e is the excentricity, and $c = \frac{e^2}{2 - e^2}$.

This series is not very commodious for arithmetical computation when e is nearly equal to 1.

THIRD SERIES, given by Euler in his *Animadversiones in Redificationum Ellipseos*, which forms a part of the second volume of his *Opuscula*.

One fourth of the perimeter is =

$$1 + (c^2 + \frac{3}{4} \cdot \frac{1}{2} c^4 + \frac{3 \cdot 5}{4 \cdot 6} \cdot \frac{1 \cdot 3}{2 \cdot 4} c^6 + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} c^8 + \&c.)$$

$$\text{hyp. log. } \frac{2}{\sqrt{c}}$$

$$- c^2 \cdot \frac{1}{4}$$

$$- \frac{3}{4} \cdot \frac{1}{2} c^4 \left(\frac{7}{12} - \frac{1}{2 \cdot 3 \cdot 4} \right)$$

$$- \frac{3 \cdot 5}{4 \cdot 6} \cdot \frac{1 \cdot 3}{2 \cdot 4} c^6 \left(\frac{13}{20} - \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{4 \cdot 6 \cdot 8} \right)$$

$$- \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} c^8 \left(\frac{19}{28} - \frac{1}{2 \cdot 3 \cdot 4} - \frac{4}{4 \cdot 5 \cdot 6} - \frac{1}{6 \cdot 7 \cdot 8} \right)$$

$$- \&c.$$

where 1 and c are the semi-transverse and semi-conjugate axes.

This

* Mr. Ivory's Paper, which contains the investigation of this series is re-published in the first part of the Mathematical Tracts and Selections.

This is a very useful series, and converges so much the faster as the excentricity is more nearly equal to 1.

FOURTH SERIES, given by Mr. Wallace, in the 5th volume of the Edinburgh Transactions.

Half the perimeter is $= \pi P (1 - eQ)$.

Where e = the excentricity, $\pi = 3.14159$

$$e' = \frac{1 - \sqrt{1 - e^2}}{1 + \sqrt{1 - e^2}}, e'' = \frac{1 - \sqrt{1 - e'^2}}{1 + \sqrt{1 - e'^2}}, e''' = \frac{1 - \sqrt{1 - e''^2}}{1 + \sqrt{1 - e''^2}}, \&c.$$

$$P = (1 + e') (1 + e'') (1 + e''') (1 + e'''), \&c.$$

$$\text{and } Q = \frac{e}{2} + \frac{e e'}{2.2} + \frac{e e' e''}{2.2.2} + \frac{e e' e'' e'''}{2.2.2.2} + \&c.$$

Or,

if α be an arc such that $\sin. \alpha = e$,

$$\text{then } e' = \tan^2 \frac{\alpha}{2}, \text{ and } 1 + e' = \sec^2 \frac{\alpha}{2}.$$

Again, by taking α' such that $\sin. \alpha' = e'$,

we have $e'' = \tan^2 \frac{\alpha'}{2}$, and $1 + e'' = \sec^2 \frac{\alpha'}{2}$; and by continuing in this manner as far as may be necessary we have

$$P = \sec^2 \frac{\alpha}{2} \cdot \sec^2 \frac{\alpha'}{2} \cdot \sec^2 \frac{\alpha''}{2}, \&c.$$

$$Q = \frac{\sin. \alpha}{2} + \frac{\sin. \alpha \cdot \sin. \alpha'}{2.2} + \frac{\sin. \alpha \cdot \sin. \alpha' \cdot \sin. \alpha''}{2.2.2} + \&c.$$

Where the values of P and Q may be computed with great facility by means of the common trigonometrical tables.

This series has the advantage of converging quickly in every case of excentricity that can occur, but more especially when it does not exceed $\sqrt{\frac{1}{2}}$.

FIFTH SERIES, by Mr. Woodhouse, inserted in the Transactions of the Royal Society of London, for the year 1804:

One fourth of the perimeter is =

$$1-b \left\{ \frac{b.(1+b)}{2.2} + \frac{b'.b.(1+b).(1+b')}{2.2.2.2} + \&c. \right\} + \frac{b}{2^m} \cdot P' \cdot Q' \cdot h.l. \frac{4}{(m)b}$$

Where 1 and b are the semi-axes,

$$b' = \frac{b^2}{1 + \sqrt{1 - b^2}}, b'' = \frac{b'^2}{1 + \sqrt{1 - b'^2}}, \&c.$$

$$P = (1 + b') (1 + b'') (1 + b''') \dots (1 + \beta).$$

$$Q = \frac{b}{2} + \frac{b}{2} \cdot \frac{b'}{2} + \frac{b}{2} \cdot \frac{b'}{2} \cdot \frac{b''}{2} + \&c.$$

and

and $\frac{1}{2}$, the term of the series $'b, ''b, ''''b, \&c.$ after which, without sensible error, each succeeding term may be taken equal to the square of the preceding term. This formula, like the preceding, is generally applicable, and particularly when the eccentricity exceeds $\sqrt{\frac{1}{2}}$.

For the investigation of these series, we must refer the reader to the respective works already cited, or to the learned memoir of the last-mentioned author, where the several investigations are given, and their mutual relation to each other pointed out.

We shall now exhibit the computation of the periphery of an ellipse, whose eccentricity is .99 and semi-axis major 1, as required by the question.

And first by the third Series.

$$e = 1 - (.99)^2 = .0199; \quad \frac{3}{8} e^2 = .00991485; \quad \frac{15}{64} e^4 = .0000019;$$

$$\frac{1}{4} e^2 = .0049750; \quad \frac{13}{24} \cdot \frac{3}{8} e^4 = .0000804; \quad \frac{3}{5} \cdot \frac{15}{64} e^6 = .0000012;$$

and hyp. log. of $\frac{2}{\sqrt{1-e^2}} = 1.6724061$. Hence

$$1 + (e^2 + \frac{3}{8} e^4 + \frac{15}{64} e^6) \times \text{h. l. } \frac{2}{\sqrt{1-e^2}} = + 1.0335324$$

$$- \frac{1}{4} e^2 - \frac{13}{24} \cdot \frac{3}{8} e^4 - \frac{3}{5} \cdot \frac{15}{64} e^6 = - 0.0050566$$

$$\text{one-fourth of the periphery} = 1.0284758$$

Multiply by 4

$$\text{And the perimeter of the ellipse is} = 4.1139032.$$

Secondly, by the fourth series.

1. Calculation of the logarithms of $e', 1 + e', 1 + e'', \&c.$

Logarithms.

$$e = .99 = \sin. \alpha = \sin. 81^\circ 53' 25'' .33 \quad 1.9956352$$

$$e' = \tan.^2 \frac{\alpha}{2} = \sin. a' = \sin. 48^\circ 49' 43'' .22 \quad 1.8766476$$

$$1 + e = \sec.^2 \frac{\alpha}{2} \quad 0.2437186$$

$$e'' = \tan.^2 \frac{a'}{2} = \sin. a'' = \sin. 11^\circ 53' 26'' .44 \quad 1.3139620$$

$$1 + e' = \sec.^2 \frac{a'}{2} \quad 0.0813636$$

$e''' =$

Logarithms.

$$e'' = \tan.^2 \frac{\alpha''}{2} = \sin. \alpha'' = \sin. 0^\circ 37' 17'' \cdot 00 \quad 2.0352326$$

$$1 + e'' = \sec.^2 \frac{\alpha''}{2} \quad 0.0546846$$

$$e^{IV} = \tan.^2 \frac{\alpha^{IV}}{2} = \sin. \alpha^{IV} = \quad 5.4684810$$

$$1 + e^{IV} = \sec.^2 \frac{\alpha^{IV}}{2} \quad 0.0000128$$

1. Computation of $\pi P(1 - eQ)$

$$\frac{e^2}{2} \quad .4900500 \quad 1.6902404$$

$$\frac{e^2 e'}{2 \cdot 2} \quad .1844412 \quad 1.2658580$$

$$\frac{e^2 e' e''}{2 \cdot 2 \cdot 2} \quad .0190016 \quad 2.2787900$$

$$\frac{e^2 e' e'' e^{IV}}{2 \cdot 2 \cdot 2 \cdot 2} \quad .0001030 \quad 4.0129926$$

$$\frac{1}{2} \quad 1. \quad (1 - eQ) = .3064042 \quad 1.4862947$$

$$1 + e' \quad - \quad - \quad - \quad - \quad 0.2437186$$

$$1 + e'' \quad - \quad - \quad - \quad - \quad 0.0813636$$

$$1 + e''' \quad - \quad - \quad - \quad - \quad 0.0046846$$

$$1 + e^{IV} \quad - \quad - \quad - \quad - \quad 0.0000128$$

$$\pi \quad - \quad - \quad - \quad - \quad 0.4971499$$

$$\pi P(1 - eQ) = 2.056952 \quad 0.3132242$$

$$\frac{2}{2} \quad \text{Perimeter} = 4.113904$$

Thirdly, by the fifth series.

$$b = \sqrt{(1 - (.99)^2)} = .1410673$$

$$b = \frac{b^2}{\left\{ 1 + \sqrt{(1 - b^2)} \right\}^2} = \frac{1}{199} = .0050251$$

$$''b = \frac{b^2}{\left\{ 1 + \sqrt{(1 - b^2)} \right\}^2} = .000006313.$$

but $\frac{b^2}{4} = .0000063129$, which differs from $''b$ only in the ninth place

place of decimals; therefore b may be taken equal $\frac{b^2}{4}$ without sensible error; and consequently $(m)b = b$, or $m = 1$.

Now

$$b \left(\frac{b}{2} + \frac{1+b}{2} \right) = \frac{b^2}{4} (1+b) = \frac{b^2}{4} \times \frac{200}{199} = .005$$

$$P = 1 + \frac{1}{2} = 1 + \frac{1}{199} = \frac{200}{199}$$

$$Q = \frac{b}{2} + \frac{b}{2} \cdot \frac{b}{2} = .0707108$$

$$\text{hyp. log. } \frac{4}{b} = \text{hyp. log. } 796 = 6.679599$$

$$\frac{b}{2} \cdot P \cdot Q \cdot \text{hyp. log. } \frac{4}{b} = .0334818.$$

Hence $1 - .005 + .0334818 = 1.0284808$, and the whole perimeter $= 1.0284808 \times 4 = 4.1139232$, nearly.

X. QUESTION 100, from the CAMB. UN. CAL.

Shew that the rectification of the hyperbola may be deduced from that of the ellipse?

FIRST SOLUTION.

Put a for the semi-transverse axis, e for the excentricity, x the abscissa, reckoned from the centre on the transverse axis, H the hyperbolic arc, reckoned from the extremities of the ordinate and transverse axis.

Then $H = x \sqrt{\left(\frac{e^2 x^2 - 1}{x^2 - 1} \right)}$, and to assign the fluent of this expression by means of elliptic arcs, put $x =$

$$\sqrt{\left(\frac{1 - z^2}{1 - e^2} \right)} \text{ and } \frac{1}{e} = m; \text{ then}$$

$$\sqrt{\left(\frac{e^2 x^2 - 1}{x^2 - 1} \right)} = \frac{(1 - m^2) z}{(1 - z^2)^{\frac{1}{2}} \sqrt{1 - m^2 z^2}}$$

Now

Now, when $z = 0$, x is $= 1$, and when $z = 1$, x is $= \infty$; therefore the fluent of $z \sqrt{\left(\frac{c^2 x^2 - 1}{x^2 - 1}\right)}$ between the values of $x = 1$ and $x = \infty$ is equal to the fluent of $\frac{(1 - m^2)z}{(1 - z^2)^{\frac{1}{2}} \sqrt{(1 - m^2 z^2)}}$ between the values of $z = 0$ and $z = 1$.

Put $z \sqrt{\left(\frac{1 - m^2 z^2}{1 - z^2}\right)} = F$;

then taking the fluxions we have

$$\dot{F} = z \sqrt{\left(\frac{1 - m^2 z^2}{1 - z^2}\right)} - \frac{(1 - m^2)z}{\sqrt{\{(1 - z^2)(1 - m^2 z^2)\}}} + \frac{(1 - m^2)z}{(1 - z^2)^{\frac{1}{2}} (1 - m^2 z^2)^{\frac{1}{2}}}$$

therefore $\dot{H} = \dot{F} - z \sqrt{\left(\frac{1 - m^2 z^2}{1 - z^2}\right)} + \frac{(1 - m^2)z}{\sqrt{\{(1 - z^2)(1 - m^2 z^2)\}}}$

But the fluent of $z \sqrt{\left(\frac{1 - m^2 z^2}{1 - z^2}\right)}$ is well known to be equal to the length of the elliptic arc whose semi-transverse axis is 1, excentricity m and abscissa z .

Call this arc E . Then

$$H = F - E + \text{fluent of } \frac{(1 - m^2)z}{\sqrt{\{(1 - z^2)(1 - m^2 z^2)\}}}$$

To find the fluent of this last term put

$$m = \frac{1 - \sqrt{(1 - m'^2)}}{1 + \sqrt{(1 - m'^2)}} \text{ and } z = \frac{2}{1 - m} \cdot z' \sqrt{\left(\frac{1 - z'^2}{1 - m'^2 z'^2}\right)}$$

then the fluent of $\frac{(1 - m^2)z}{\sqrt{\{(1 - z^2)(1 - m^2 z^2)\}}}$

$$= 2mz - \text{fluent of } 2(1 + m)z' \sqrt{\left(\frac{1 - m'^2 z'^2}{1 - z'^2}\right)} + 2E.$$

But the fluent of $z' \sqrt{\left(\frac{1 - m'^2 z'^2}{1 - z'^2}\right)}$ is the elliptic arc E' whose semi-transverse axis is 1, excentricity m' and abscissa z' .

$$\text{Hence } H \begin{cases} = F + 2mz - 2(1 + m)E' + E \\ = z \sqrt{\left(\frac{1 - m^2 z^2}{1 - z^2}\right)} + 2mz - 2(1 + m)E' + E. \end{cases}$$

So that the length of an hyperbolic arc is ascertained by means of two elliptic arcs and an algebraical quantity.

This discovery was first made by Mr. Landen, and published in the Philosophical Transactions for 1775, and afterwards in his Memoirs*; but the substance of the above solution is taken from Mr. Woodhouse's Paper, referred to in the preceding solution.

SECOND SOLUTION, by M. C. BOSSUT.

Let a denote the semi-transverse axis, and b the semi-conjugate axis of an ellipse; and E an arch of this ellipse intercepted between the conjugate axis and an ordinate parallel thereto at the distance x from the centre.

Then it is well known that $E = \arcsin \sqrt{\frac{1 - (1 - b^2)x^2}{1 - x^2}}$.

Put $1 - (1 - b^2)x^2 = u^2$, then $x = \sqrt{\frac{1 - u^2}{1 - b^2}}$; and

$$E = \frac{\arcsin u}{\sqrt{\{(1 + b^2)u^2 - u^4 - b^2\}}}. \text{ Let } \sqrt{\{(1 + b^2)u^2 - u^4 - b^2\}} = y,$$

and the last equation will be transformed into

$$E = -\frac{y}{2} - \frac{(1 + b^2 - y^2)y}{\sqrt{\{(1 + b^2 - y^2)^2 - 4b^2\}}}, \text{ which may be further}$$

abridged by putting $1 + b^2 = c^2$, $1 + b = \tau$, $1 - c = \pi$; for then

$$E = -\frac{y}{2} - \frac{(c^2 - y^2)y}{2\sqrt{(\tau^2 - y^2)} \cdot \sqrt{(\pi^2 - y^2)}}. \text{ Let us now assume}$$

$$\frac{c^2 - y^2}{\sqrt{(\tau^2 - y^2)} \cdot \sqrt{(\pi^2 - y^2)}} = \frac{A\sqrt{(\tau^2 - y^2)}}{\sqrt{(\pi^2 - y^2)}} + \frac{B\sqrt{(\pi^2 - y^2)}}{\sqrt{(\tau^2 - y^2)}}.$$

By means of this equation $A + B = 1$, $A\tau^2 + B\pi^2 = c^2$, and consequently $A = \frac{c^2 - \pi^2}{\tau^2 - \pi^2}$, $B = \frac{\tau^2 - c^2}{\tau^2 - \pi^2}$; so that

$$E = -\frac{y}{2} - \frac{A y \sqrt{(\tau^2 - y^2)}}{2\sqrt{(\pi^2 - y^2)}} - \frac{B y \sqrt{(\pi^2 - y^2)}}{2\sqrt{(\tau^2 - y^2)}}; \text{ or}$$

$$E = -\frac{y}{2} - \frac{A}{2} \int \frac{y \sqrt{(\tau^2 - y^2)}}{\sqrt{(\pi^2 - y^2)}} - \frac{B}{2} \int \frac{y \sqrt{(\pi^2 - y^2)}}{\sqrt{(\tau^2 - y^2)}}.$$

* Mr. Landen's Memoir is reprinted in the early numbers of the Old Series of the Repository.

In order to find the fluent of $\frac{y\sqrt{x^2-y^2}}{\sqrt{x^2-y^2}}$, put $x^2-y^2=z^2$;

$$\text{then } \int \frac{y\sqrt{x^2-y^2}}{\sqrt{x^2-y^2}} = - \int \frac{z^2 \dot{z}}{\sqrt{x^2-z^2} \cdot \sqrt{\{z^2-(x^2-x^2)\}}},$$

which is known to represent the arch of an ellipse intercepted between the conjugate axis and an ordinate parallel thereto, at the distance $\frac{x\sqrt{x^2-z^2}}{\sqrt{x^2-x^2}}$ from the centre.

Call this arch E'.

And to find the fluent of $\frac{y\sqrt{x^2-y^2}}{\sqrt{x^2-y^2}}$, put $x^2-y^2=t^2$;

$$\text{then } \int \frac{y\sqrt{x^2-y^2}}{\sqrt{x^2-y^2}} = \int - \frac{t^2 \dot{t}}{\sqrt{x^2-t^2} \cdot \sqrt{t^2+x^2-t^2}} = \\ \frac{\sqrt{x^2-t^2} \cdot \sqrt{t^2+x^2-t^2}}{t} - \int \frac{-x^2(t^2-x^2)\dot{t}}{t^2\sqrt{x^2-t^2} \cdot \sqrt{t^2+x^2-t^2}}.$$

But the fluent of $\frac{-x^2(t^2-x^2)\dot{t}}{t^2\sqrt{x^2-t^2} \cdot \sqrt{t^2+x^2-t^2}}$ expresses the arch of an hyperbola of which the semi-axes are x and $\sqrt{x^2-x^2}$, and abscissa, reckoned from the centre, on the axis x , $\frac{x^2\sqrt{t^2+x^2-x^2}}{x t}$.

Call this arch H.

Hence it appears, that

$$E = -\frac{y}{2} = \frac{A}{2} \times E' - \frac{B}{2} \times \frac{\sqrt{x^2-t^2} \cdot \sqrt{t^2+x^2-t^2}}{t} + \frac{B}{2} \times H;$$

and, consequently,

$$H = \frac{2E}{B} + \frac{y}{B} + \frac{A \times E'}{B} + \frac{\sqrt{x^2-t^2} \cdot \sqrt{t^2+x^2-t^2}}{t}.$$

XI. QUESTION 101, from the CAMB. UN. CAL.

In the lemniscata, whose equation is $(x^2+y^2)^2 = x^2-y^2$, it is required to assign two arcs equal to one another.

FIRST SOLUTION, by Mr. WALLACE, R. M. College.

1. Let CPA (fig. 171, pl. 9,) be the lemniscata, C its centre, CA its semi-axis, and PB an ordinate at any point P in the curve; then $CB = x$, $PB = y$. Join CP and put r for the variable line CP, and a for the semi-axis CA, then, the given equation $(x^2 + y^2)^2 = x^2 - y^2$ may be expressed thus,

$$r^2 = a \sqrt{x^2 - y^2}.$$

Put ϕ for the variable angle PCA, then, because $x = r \cos \phi$ and $y = r \sin \phi$, this last equation becomes $r^2 = ar \sqrt{(\cos^2 \phi - \sin^2 \phi)}$, which, by substituting $\cos 2\phi$ for its equal $\cos^2 \phi - \sin^2 \phi$, and dividing by r becomes

$$r = a \sqrt{(\cos 2\phi)} \dots \dots \dots (1.)$$

Draw C q indefinitely near to CP, and pq perpendicular to CP, then, if z be put for AP, the arch of the curve, it is manifest that $\dot{z} = \sqrt{(Pq^2 + pq^2)}$.

But Pq is the fluxion of r , or of $a \sqrt{(\cos 2\phi)}$, which is $-\frac{a \phi \sin 2\phi}{\sqrt{(\cos 2\phi)}}$, and pq is equal to $r \phi$, that is to $a \phi \sqrt{(\cos 2\phi)}$, therefore substituting for Pq , and pq , these values and putting 1 for $\sin^2 2\phi + \cos^2 2\phi$, we have

$$\dot{z} = \frac{a \phi}{\sqrt{(\cos 2\phi)}} \dots \dots \dots (2)$$

This fluxionary expression, although very simple in its form, does not admit of an algebraic integral, nor even of one involving circular arcs or logarithms only. The integral may however be expressed in a finite form by means of two elliptic arcs; but the indefinite rectification of the curve is not the object of our present enquiry.

2. Leaving for the present the consideration of the lemniscata, let us suppose that ϕ and ψ are two angles so related to each other that the product of their co-sines, or $\cos \phi \cos \psi$ is a constant number, then, taking the fluxion of the product, we shall have

$$\phi \sin \phi \cos \psi + \psi \sin \psi \cos \phi = 0.$$

Let us next suppose that the constant number is the cosine of 45° , or that

$$\cos \phi \cos \psi = \sqrt{\frac{1}{2}}$$

then, by taking the squares of each side of this equation and putting $1 - \sin^2 \phi$ for $\cos^2 \phi$ we get

$$(1 - \sin^2 \phi) \cos^2 \psi = \frac{1}{2}$$

and

and $\frac{1}{2} (2 \cos.^2 \psi - 1) = \sin.^2 \phi \cos.^2 \psi$

but $2 \cos.^2 \psi - 1 = \cos. 2\psi$

therefore $\sqrt{(\frac{1}{2} \cos. 2\psi)} = \sin. \phi \cos. \psi$,

and, since ϕ and ψ enter precisely alike into the assumed equation $\cos. \phi \cos. \psi = \sqrt{\frac{1}{2}}$ we have also

$$\sqrt{(\frac{1}{2} \cos. 2\psi)} = \sin. \psi \cos. \phi.$$

Let these values of $\sin. \phi \cos. \psi$ and $\sin. \psi \cos. \phi$ be substituted in the last fluxionary equation and it becomes

$$\dot{\phi} \sqrt{(\frac{1}{2} \cos. 2\psi)} + \dot{\psi} \sqrt{(\frac{1}{2} \cos. 2\phi)} = 0$$

$$\text{and hence } \frac{\dot{\phi}}{\sqrt{(\cos. 2\phi)}} + \frac{\dot{\psi}}{\sqrt{(\cos. 2\psi)}} = 0 \dots\dots\dots (3)$$

Hence we learn that whatever values be given to the angles ϕ and ψ , provided the product of their cosines is $\sqrt{\frac{1}{2}}$, the sum

of the fluents of the two expressions $\frac{\dot{\phi}}{\sqrt{(\cos. 2\phi)}}$ and $\frac{\dot{\psi}}{\sqrt{(\cos. 2\psi)}}$

under whatever form they may be exhibited, or that sum multiplied by any constant quantity, must necessarily be a constant quantity.

3. Let us now recur to the lemniscata and suppose that CP, CQ, are two straight lines drawn from the centre meeting the curve in P and Q, and that the angles PCA, QCA which we shall denote by ϕ and ψ respectively, are such that the product of the cosines is equal to $\sqrt{\frac{1}{2}}$ or to the cosine of half a right angle, then as from equation (2) the fluxion of the arch AP will

be $\frac{a\dot{\phi}}{\sqrt{(\cos. 2\phi)}}$ and that of the arch AQ $\frac{a\dot{\psi}}{\sqrt{(\cos. 2\psi)}}$ it is mani-

fest that the sum of the arches AP, AQ, will have the same property as the sum of the fluents of the two quantities which constitute the fluxionary equation (3), that is, *the sum of the arches AP, AQ, will be equal to a constant quantity.*

To determine this quantity we have only to give determinate values to the angles ϕ and ψ , and the constant quantity will evidently be the sum of the arches of the lemniscata corresponding to these values. Let us therefore assume $\psi = 0$, and consequently $\cos. \psi = 1$, then from the equation $\cos. \phi \cos. \psi = \cos. 45^\circ$ we have $\phi = 45^\circ$; now when $\psi = 0$, the arch of the curve corresponding to ϕ vanishes, and when $\phi = 45^\circ$, it appears from the equation (1) which expresses the nature of the curve that the corresponding arch of the lemniscata is its quadrant APC. Therefore the constant quantity to which the sum of the arches AP, AQ is always equal is the quadrant APQC.

This property of the curve is very remarkable, and furnishes directly

directly a solution to the question; for since the quadrant APQC is equal to the sum of the arches AQ, AP, from each take away the common arch AP and the arch AQ reckoned from the one extremity of the semi-axis will be equal to the arch CP reckoned from the other extremity. This property of the curve which answers the question may be stated in the form of a theorem: thus,

Let C be the centre and CA the semi-axis of the lemniscata; from C let there be drawn two chords CP, CQ, meeting the curve in P and Q, and let the angles ACP, ACQ, be such that the product of their cosines is equal to the cosine of half a right angle, then the arches of the curve, reckoned from the points P, Q, to the nearest extremities of the axis, are equal to each other.

This theorem, which is analogous to those relating to arches of conic sections given in Art. 3, Part II. Vol. I. of the Repository is capable of being rendered more general.

If we suppose the arches ϕ and ψ equal to each other, then the points P and Q will coincide at a point R; let RC be the position in which the lines CP, CQ, coincide and let θ denote the angle RCA, then as $\text{cof. } \phi \text{ cof. } \psi$ becomes $\text{cof.}^2 \theta$ we have $\text{cof.}^2 \theta = \sqrt{\frac{1}{2}}$, therefore the angle θ is given, and consequently the point R, that is the middle of the quadrantal arch CRQ, is given. Thus it appears that the quadrant of the lemniscata like the quadrant of a circle admits of being bisected geometrically. It may also like the circle be divided into any number of equal parts by the resolution of equations, but I shall not prosecute this subject at present any further.

SECOND SOLUTION, by Mr. IVORY, R. M. College.

The equation of the lemniscata will be satisfied by putting

$$x^2 = \frac{1}{2} (z^2 + z^4)$$

$$y^2 = \frac{1}{2} (z^2 - z^4)$$

that is, by putting $x = \frac{z}{\sqrt{2}} \sqrt{1 + z^2}$

$$y = \frac{z}{\sqrt{2}} \sqrt{1 - z^2}$$

Let w be the arch of the curve corresponding to the co-ordinates x and y : then

$$w = \sqrt{(x^2 + y^2)} = \frac{1}{\sqrt{2}} \times z \sqrt{\left\{ \frac{(1+z^2)^2}{1+z^2} + \frac{(1-z^2)^2}{1-z^2} \right\}} = \frac{z}{\sqrt{1-z^4}}$$

And

And if x' and y' be any other two co-ordinates, and w' the corresponding arch of the curve; and if

$$x' = \frac{z'}{\sqrt{z}} (1 + z^2)$$

$$y' = \frac{z'}{\sqrt{z}} \sqrt{(1 - z'^2)}$$

then in like manner $w' = \frac{z'}{\sqrt{(1 - z'^2)}}$.

Suppose now that z and z' are so related that

$$z'^2 = \frac{1 - z^2}{1 + z^2} \text{ or } z' = \sqrt{\left(\frac{1 - z^2}{1 + z^2}\right)}; \text{ then } \dot{z}' = \frac{-\dot{z}}{\sqrt{(1 - z^2)}} \times \frac{2z}{1 + z^2},$$

$$\text{and } \sqrt{(1 - z'^2)} = \frac{2z}{1 + z^2}; \text{ therefore } \frac{\dot{z}'}{\sqrt{(1 - z'^2)}} = \frac{-\dot{z}}{\sqrt{(1 - z^2)}};$$

that is, $\dot{w}' = -\dot{w}$, or $\dot{w}' + \dot{w} = 0$; and taking the fluents $w' + w = C$, where C is a constant quantity.

From the equation of the lemniscata it is apparent, that while x^2 increases from 0 to 1, y^2 first increases to a maximum value, and then decreases to 0. Therefore taking all possible values of $+x$, between the limits 0 and $+1$, upon an axis, there will correspond to each value of $+x$, a double ordinate $\pm y$, one on one side of the axis, and one on the other side of it: and thus an oval shaped figure will be completed on one side of the origin of x . And, in like manner, taking all possible values of $-x$, between the limits 0 and -1 , upon the same axis, an equal and similar oval shaped figure will be formed on the other side of the origin of x . Thus the lemniscata consists of two equal and similar oval shaped curves, joined together in the origin of x , resembling the numerical character 8, as the figure $AnmB\mu\nu ACA$ (fig. 172, pl. 9.)

Let $Aq = x$, $qn = y$; also $Ap = x'$ and $pm = y'$: then $w = \text{arch } An$, and $w' = \text{arch } Am$: and the equation $w' + w = C$, shews that $\text{arch } An + \text{arch } Am$ is a constant quantity. The relation between z and z' shews that $z' = 1$ when $z = 0$; and hence $x' = 1$, $y' = 0$, when $x = 0$ and $y = 0$: therefore the arch Am is equal to the semi-oval $AnmB$, when the arch An is equal to 0. Hence it is obvious that the semi-oval $AnmB$ is equal to the constant quantity C . Therefore the two arcs An and Am are constantly equal to the semi-oval $AnmB$. Therefore the arch An is equal to the arch Bm ; and thus an infinite number of equal arches may be assigned on the lemniscata, reckoned from opposite ends of the semi-oval,

Since

Since $z^2 = \frac{1-z^2}{1+z^2}$, therefore $z^2 = \frac{1-z'^2}{1+z'^2}$; but $z^2 = x^2 + y^2$, and $\frac{y'^2}{x'^2} = \frac{1-z'^2}{1+z'^2}$: therefore $x^2 + y^2 = \frac{y'^2}{x'^2}$. Hence

this geometrical construction for finding equal arcs on the lemniscata: draw BD perpendicular to AB and take BO = the chord An, then join AO cutting the curve in m; and the arch An is equal to the arch Bm.

THIRD SOLUTION, by P. Q. of C. C. C.

Put v for $x^2 + y^2$, then by the question v^2 will be equal to $x^2 - y^2$, and consequently

$$x = \frac{1}{\sqrt{2}} \sqrt{(v + v^2)}, \quad \dot{x} = \frac{\dot{v}}{\sqrt{2}} \cdot \frac{\frac{1}{2} + v}{\sqrt{(v + v^2)}}$$

$$y = \frac{1}{\sqrt{2}} \sqrt{(v - v^2)}, \quad \dot{y} = \frac{\dot{v}}{\sqrt{2}} \cdot \frac{\frac{1}{2} - v}{\sqrt{(v - v^2)}}$$

$$\text{and } \sqrt{(\dot{x}^2 + \dot{y}^2)} = \frac{\dot{v}}{2\sqrt{v}\sqrt{(1-v^2)}}.$$

Now we are told by the very ingenious Mr. Landen, in the first Theorem of his 111 Table of Fluents given at the end of his Mathematical Memoirs, (see also vol. 1. p. 60, of the Mathe-

matical Repository, Old Series) that the fluent of $\frac{\dot{v}}{\sqrt{v}\sqrt{(1-v^2)}}$

generated whilst v from 0 becomes equal to any quantity c is equal to the fluent of the same fluxion generated whilst v from $\frac{1-c}{1+c}$ becomes equal to 1. From this property of the fluent we

immediately derive a solution of the question. For the figure of the lemniscata being supposed known, let C (fig. 173, pl. 9.) be the point in which the curve intersects itself, and CA its semi-axis, which in the question is supposed = 1, also CD and DP any abscissa and its corresponding ordinate, then the square of the chord drawn from C to P will represent v : let us now suppose that another chord CP' is drawn, and that the two chords CP, CP', are so related to each other that CP being put equal

to c , CP' may be equal to $\sqrt{\left(\frac{1-c^2}{1+c^2}\right)}$, then it is obvious

that

that the arches CP, PA, may be considered as fluents generated according to the conditions of the theorem, they will therefore be equal to each other, and in this way may any number of equal arches be found agreeably to what is required in the question.

It perhaps may not be amiss to observe, that the first theorem of Mr. Landen's table (see as before his Memoirs, or the 1st. vol. of the Mathematical Repository) exhibits the rectification of any arch of the lemniscata by means of arches of two ellipses, and the third and fourth theorems give the rectification of the whole curve; and it is particularly worthy of remark, that the whole curve may be rectified by means of the perimeters of the circle and of one ellipse only, namely, that which has its eccentricity equal to half its conjugate axis.

XII. QUESTION 102, *from the CAMB. UN. CAL.*

On a given straight line, perpendicular to the horizon, describe a circle, and draw a tangent at its higher extremity; if any point be assumed in this tangent, and a straight line be drawn from it parallel to the diameter, cutting the circle in two points, and chords be drawn from those points perpendicular to the diameter, the times of describing the parts of this line, together with the times of describing the corresponding chords with the velocities acquired at their extremities shall be equal?

FIRST SOLUTION, *by Mr. BAZLEY.*

Let AB (fig. 174. pl. 9.) be the vertical diameter of the circle AFBK; draw the tangent AG and GEF parallel to AB cutting the circle in E and F; draw the chords EI, FK, parallel to AG, cutting the diameter in C and D. Put $g = 16\frac{1}{2}$ feet; then by the laws of falling bodies, we have, the time of describing GE or AC = $\sqrt{(AC \div g)}$, and the velocity acquired at C or E = $2\sqrt{(g \times AC)}$. Now EI = $2\sqrt{(AC \times CB)} = 2\sqrt{(AC \times AD)}$, and the time of describing it, by uniform motion, is $2\sqrt{(AC \times AD)} \div 2\sqrt{(g \times AC)} = \sqrt{(AD \div g)}$, which is evidently the time of describing AD; therefore the velocity at D or F = $2\sqrt{(g \times AD)}$. Again, since FK = EI = $2\sqrt{(AC \times AD)}$, the time of describing FK, by uniform motion, is = $2\sqrt{(AC \times AD)} \div$

$2\sqrt{g \times AD} = \sqrt{AC \div g}$, which is evidently the time of describing AC. Hence the time of describing AC or GE, together with the time of describing EI is equal to the time of describing AC, together with the time of describing AD; that is, equal to the time of describing AD or GF, together with the time of describing FK.

SECOND SOLUTION, by Mr. W. MELLOR, *London.*

Put $EG = x$, $FG = y$, and $g = 16\frac{1}{2}$; then, by the property of the circle $EI = FK = 2\sqrt{xy}$. Now the time of describing the distance x is $\sqrt{x \div g}$, and the velocity is then $= 2\sqrt{gx}$; wherefore the time of describing EI with this velocity is $= \sqrt{xy \div gx} = \sqrt{y \div g}$; hence the whole time of describing $EG + EI$ is $= \sqrt{x \div g} + \sqrt{y \div g}$. Again the time of describing the distance y is $= \sqrt{y \div g}$, and the velocity is then $= 2\sqrt{gy}$. Wherefore, the time of describing the chord FK with this velocity is $= \sqrt{xy \div gy} = \sqrt{x \div g}$. Or the whole time of describing $GF + FK$ is $= \sqrt{x \div g} + \sqrt{y \div g}$; the very same as before.

XIII. QUESTION 103, from the CAMB. UN. CAL.

Let the compressive force be supposed, proportional to the square of the density, and the force of gravity inversely proportional to the square of the distance from the earth's centre, it is required to find the law of density in the atmosphere?

SOLUTION, by Mr. BAZLEY.

To render the solution as perspicuous as may be, let C (fig. 175, pl. 9.) be the earth's centre, CA its radius, and conceive CAS to be the axis of a cylinder of the atmosphere whose base is unity and its height indefinite. Put $CA = r$, $CE = x$, the density at E $= y$, the density at A $= d$, and let p denote the height of a column of air of the uniform density d , and weight equal to a column of mercury upon the same base 29 $\frac{1}{2}$ inches high.

Now

Now let the curve BG be of such a nature that the ordinate GE may represent the density at any point E; draw another ordinate IF indefinitely near to GE; and if two sections of the cylinder perpendicular to CS be conceived to pass through GEFI, it is plain that the quantity of air included between these sections is $y\dot{x}$, that is, the magnitude into the density, and this

drawn into the force of gravity at E gives — $\frac{r^2 y \dot{x}}{x^2}$ for the flux-

ion of the weight or pressure, which is negative, because the weight or pressure decreases as x increases. But by hypothesis, the pressure is as the square of the density, and consequently

— $\frac{r^2 y \dot{x}}{x^2} : 2y\dot{y}$ is a constant ratio, which, by the doctrine of flux-

ions, is the same as the ratio of their fluents; and, at the earth's surface A, this ratio is $pd : d^2$, that is as the weight to the square of the density.

Hence, $pd : d^2 :: -\frac{r^2 y \dot{x}}{x^2} : 2y\dot{y}$, and therefore $2p\dot{y} = -\frac{dr^2 \dot{x}}{x^2}$;

and the fluents are $2py = \frac{dr^2}{x}$: but when $y = d$, $x = r$,

therefore, by correction, $2p(d - y) = dr - \frac{dr^2}{x}$; whence,

$$y = d - \frac{dr}{2p} + \frac{dr^2}{2px}.$$

By following the same method, the law of density upon any other hypothesis of gravity and compression may be readily found:—Thus, if the density be as the compression and the force of gravity uniform, we have, $pd : d :: -y\dot{x} : \dot{y}$, whence,

$p\dot{y} = -y\dot{x}$, or $-\frac{p\dot{y}}{y} = \dot{x}$; and the fluents are $p \times \text{hyp. log.}$

$\frac{1}{y} = x$; and, by correction, $p \times \text{hyp. log. } \frac{d}{y} = x - r$, which

agrees with the rule usually given by writers on pneumatics.

XIV. QUESTION 104, *from the CAMB. UN. CAL.*

A body revolving in a given circle, acted on by a force, which is inversely proportional to the square of the distance from the centre, in consequence of an impulse in the direction of its motion, begins to describe an orbit of a given excentricity; find the velocity communicated to it by the impulse?

FIRST SOLUTION, *by Mr. BAZLEY.*

Let S (fig. 176, pl. 9.) be the centre of force, A the point of impact. Since the direction of the body's motion at A is perpendicular to the axis AB, and its velocity (by the question,) greater than that wherewith it described a circle at the same distance SA; it is plain, from the laws of central forces, that A is the lower apse. Let F be the other focus, C the centre, V the velocity in the circle. Then, by Emerson's centripetal forces, prop. 14, cor. 1, $\sqrt{FA} : \sqrt{CA} :: \text{vel. in the ellipse} : \text{vel. in the circle} = V$; hence, the velocity in the ellipse at A is $= \sqrt{\{FA \div CA\} \times V}$; where FA and CA are given, because SA and CS are given. Whence the velocity communicated is

$$\sqrt{\left(\frac{FA}{CA} \times V\right)} - V = \frac{\sqrt{FA} - \sqrt{CA}}{\sqrt{CA}} \times V.$$

SECOND SOLUTION, *by Mr. R. SIMPSON.*

Let S be the centre, and SA the radius of the given circle; then, since the body is acted on by a force which is inversely proportional to the square of the distance from the centre, the orbit described by the body must be a conic section. Let SF be taken on AS produced equal to twice the given excentricity, and make FB = AS; then, because the body, when at A, is impelled by a force acting perpendicularly to AS, it is obvious that AB will be the transverse axis of the section, and A will be the place where the body is nearest to the centre, and B the place where the body is at the greatest distance from the centre. Now put v for the velocity in the circle at A, and V for that in the conic section, r = the radius AS, and p = the parameter, which is given, because AS and the excentricity are given. Then, by prop. xvi. cor. 3, Principia, $V : v :: \sqrt{p} : \sqrt{2r}$; or $V = v \sqrt{p \div 2r}$; and therefore $V - v = v \left\{ \sqrt{p \div 2r} - 1 \right\}$, the velocity required.

Or, if e be put for the excentricity, we have

$$V - v = v \left\{ \sqrt{\left(\frac{2e - r}{e + r}\right)} - 1 \right\}.$$

XV. QUESTION 105, from the CAMB. UN. CAL.

Two bodies whose weights are known, lying on a smooth horizontal plane, are connected by a flexible line, passing through a small ring fixed at a given point between them; in this position a given velocity is communicated to one of them in a direction perpendicular to the line that joins their centres, and the other body is made to move directly towards the ring: investigate the motion of the projected body, and find the angle described when the other body arrives at the ring.

FIRST SOLUTION, by Mr. CUNLIFFE.

Let *A* and *B* (fig. 177, pl. 9.) represent the two bodies at the commencement of motion, *AB* the line or string connecting them passing through the ring at *C*. Let the body *B*, be that to which the given velocity of *f* feet per second is communicated in a direction perpendicular to *CB*. Let the curve *BB'* represent the track of the body *B*, the point *b*, another position of *B*, indefinitely near to *B'*: join *CB'*, *Cb* and with the centre *C* and radius *CB'* describe a small arch *B'd* cutting *Cb* in *d*.

Put *CB* = *a*, *CB'* = *s*, then *db* = *s*; also let *z* denote the arch of a circle (radius *a*) intercepted between *CB*, *CB'*. Let the velocity of *B* at *B'* in the direction *CB'* be denoted by *v*, and let the time of describing *BB'* be denoted by *t*; the weights of the bodies *A* and *B* being denoted by *A* and *B* respectively.

Then we shall obviously have $a : z :: s : \frac{s z}{a} = B'd$, and

$$db : B'd :: v : \frac{B'd \times v}{db} = \frac{vsz}{as} = \text{the velocity in the direction}$$

B'd; therefore $\frac{v^2 s z^2}{a^2 s^2}$ is the centrifugal force of the body

B at *B'*.

Again, $db : B'b :: v : \frac{B'b \times v}{db} = \text{the velocity of the body } B$
in the direction of the curve *B'b*, and by the known property of
tangents $B'b : B'd :: BC : \frac{B'C \times B'd}{B'b} = \text{the length of a per-}$
pendicular

The centripetal force f acting in the direction of the radius vector may be resolved into two other forces; viz. one $= \frac{x}{r} \times f$, acting parallel to x , and tending to diminish x ; and another $= \frac{y}{r} \times f$, acting parallel to y , and tending to diminish y : but the forces tending to diminish x and y , are measured by $-\frac{\ddot{x}}{r}$, and

$-\frac{\ddot{y}}{r}$; therefore, we have these two equations $-\frac{\ddot{x}}{r} = \frac{x}{r} \times f$, and

$-\frac{\ddot{y}}{r} = \frac{y}{r} \times f$: or putting $\frac{f}{r} \times \frac{\omega'}{\omega}$ for f ,

$$\left. \begin{aligned} \frac{\ddot{x}}{r} + \frac{\omega'}{\omega} \times \frac{x}{r} \times \frac{f}{r} &= 0 \\ \frac{\ddot{y}}{r} + \frac{\omega'}{\omega} \times \frac{y}{r} \times \frac{f}{r} &= 0 \end{aligned} \right\} \dots\dots\dots (1)$$

Let the first of these equations be multiplied by $2\dot{x}$, and the second by $2\dot{y}$; then, adding them together, and remarking that $\dot{r}^2 = \dot{x}^2 + \dot{y}^2$ and $2\dot{r}\ddot{r} = 2x\ddot{x} + 2y\ddot{y}$, we get

$$\frac{2x\ddot{x} + 2y\ddot{y}}{r} + \frac{\omega'}{\omega} \times \frac{2\dot{r}\dot{r}}{r} = 0$$

and, taking the fluents,

$$\frac{\dot{x}^2 + \dot{y}^2}{r} + \frac{\omega'}{\omega} \times \frac{\dot{r}^2}{r} = c.$$

c being a constant quantity. To determine c , I remark that, at

the commencement of the motion, or when $\tau = 0$, that $\frac{\dot{r}}{r} = 0$;

for the body is projected from rest in a direction perpendicular to the radius vector, and, in the first instant of time, its velocity

to or from the ring is $= 0$: therefore, when $\tau = 0$, $\frac{\dot{x}^2 + \dot{y}^2}{r} =$

c : but $\frac{\dot{x}^2 + \dot{y}^2}{r}$ is manifestly the square of the velocity in the

trajectory, and therefore $= b^2$, when $\tau = 0$: therefore $b^2 = c$: The last equation, then, is

$$\dot{x}^2 +$$

$$\frac{\dot{x}^2 + \dot{y}^2}{\tau^2} + \frac{\omega'}{\omega} \times \frac{\dot{\epsilon}^2}{\tau^2} = b^2 \dots \dots \dots (2)$$

Multiply the first of the equations (1) by x , and the second by y ; then having added them together, we get

$$\frac{x\ddot{x} + y\ddot{y}}{\tau^2} + \frac{\omega'}{\omega} \times \frac{\epsilon\ddot{\epsilon}}{\tau^2} = 0.$$

And this equation being added to the equation (2), there results

$$\frac{\dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y}}{\tau} + \frac{\omega'}{\omega} \times \frac{\dot{\epsilon}^2 + \epsilon\ddot{\epsilon}}{\tau^2} = b^2$$

But $\dot{\epsilon}\ddot{\epsilon} = x\ddot{x} + y\ddot{y}$ and $\dot{\epsilon}^2 + \epsilon\ddot{\epsilon} = \dot{x}^2 + \dot{y}^2 + x\ddot{x} + y\ddot{y}$, therefore by substitution,

$$\frac{\dot{\epsilon}^2 + \epsilon\ddot{\epsilon}}{\tau} = \frac{\omega}{\omega + \omega'} \times b^2 \tau.$$

And, taking the fluents

$$\frac{\epsilon\dot{\epsilon}}{\tau} = \frac{\omega}{\omega + \omega'} \times b^2 \tau$$

no constant quantity being necessary because $\frac{\epsilon}{\tau} = 0$, when $\tau = 0$.

And taking the fluents again

$$\epsilon^2 = a^2 + \frac{ab^2}{\omega + \omega'} \times \tau^2$$

$$\text{or } \epsilon = a \times \sqrt{1 + \frac{\omega}{\omega + \omega'} \times \frac{b^2}{a^2} \tau^2}.$$

Again multiply the first of the equations (1) by y , and the second by x , and subtract the one from the other, then

$$\frac{y\ddot{x} - x\ddot{y}}{\tau^2} = 0.$$

And taking the fluents

$$\frac{y\dot{x} - x\dot{y}}{\tau} = c$$

c being a constant quantity. To determine c , it is obvious that $y\dot{x} - x\dot{y}$ is the small area described by the radius vector in the time τ , which area, when $\tau = 0$, is manifestly $= a \times b \times \tau$.

Therefore $c = a \times b$: and

$$\frac{y\dot{x} - x\dot{y}}{r} = ab,$$

Let ϕ = angle that r makes with the line from which the motion commences (reckoned on the periphery of the circle whose radius = 1,) then $x = r \sin. \phi$ and $y = r \cos. \phi$: And the last equation becomes, by substitution,

$$r^2 \dot{\phi} = ab \tau,$$

therefore $\dot{\phi} = \frac{ab\tau}{r^2} = \frac{b}{a} \times \frac{\tau}{1 + \frac{\omega}{\omega + \omega'} \times \frac{b^2}{a^2} \tau^2}$: hence

$$\phi = \frac{\omega + \omega'}{\omega} \times \text{arc. tan. } \frac{b}{a} \tau \sqrt{\frac{\omega}{\omega + \omega'}}.$$

XVI. QUESTION 106, from the CAMB. UN. CAL.

If a body, suspended by a string, oscillates through a quadrant (the extremity of the quadrant being the lowest point,) the force stretching the string at the lowest point is three times that which is due to the weight of the body. Required proof?

FIRST SOLUTION, by Mr. BAZLEY.

Put r = radius of the quadrant, and $g = 16\frac{1}{2}$ feet; and if the force of gravity be measured by $2g$, we shall have the centrifugal force = $v^2 \div r$; but at the lowest point of the quadrant the velocity v is $2\sqrt{gr}$, and the centrifugal force there = $v^2 \div r = 4g$. Hence $2g : 4g :: W$, the weight of the body : $2W$ = the tension of the string due to the centrifugal force; to which add the weight of the body itself, and we have the whole tension of the string = $3W$.

SECOND SOLUTION, by Mr. JOHNSON, Birmingham.

Let C, (fig. 178, pl. 9.) be the point of suspension, and ADB the quadrant described by the oscillating body, AC being parallel, and BC perpendicular to the horizon. Then to find the force which stretches the string at any point D, put $r = AC$, or the length of the string; w = the weight of the body; $f = 32\frac{1}{2}$ feet,

feet, the force of gravity; a = the angle ACD described by the body; v = the velocity at D; and draw DE perpendicular to AC. Then, the force acting on the string at D arises jointly from the centrifugal force in the direction DC, and the force of gravity in the same direction: but the centrifugal force at D is to the force of gravity as $v^2 \div r$ to 1, and the velocity in descending through the arc AD is equal to the velocity in descending through DE, or $v = \sqrt{2g \cdot DE} = \sqrt{2gr \sin. a}$; therefore, $v^2 \div r = 2g \sin. a$; or the centrifugal force is to the force of gravity as $2 \sin. a$ to 1; and, consequently, the effect of the centrifugal force at D in the direction DC is equal to $w \times 2 \sin. a$. Again, since the force of gravity acts in the direction ED, its effect in the direction DC will be equal to $w \times \sin. a$; and therefore the whole force stretching the string will be equal to $w \times 3 \sin. a$. But when the body arrives at B, $\sin a = 1$; and, consequently, the whole force at the lowest point is $= 3w$.

XVII. QUESTION 107, from the CAMB. UN. CAL.

Let two weights fastened to two equal strings, (l, l) move in a vertical plane, the one oscillating, the other revolving; and at the lowest points let the velocities due to the heights h, h' , be

such, that $\frac{h}{2l} = \frac{2l}{h'} = m^2$, then the time of the revolution of

the one weight, is to the time of the oscillation of the other, as m to 1. Required proof?

SOLUTION, by Mr. CUNLIFFE.

In the case of the revolving weight $\frac{lx}{2g^{\frac{1}{2}} \sqrt{(h'-x)} \sqrt{(2lx-x^2)}}$

will express the fluxion of the time; and in the case of the

oscillating weight $\frac{lv}{2g^{\frac{1}{2}} \sqrt{(h-v)} \sqrt{(2lv-v^2)}}$ will express the

fluxion of the time of oscillation. Now what is required is the ratio of the fluent of the former expression, generated whilst x from 0 becomes $2l$, to the fluent of the latter expression, generated

rated whilst v from 0, becomes h : or the ratio of the fluent

$\frac{x}{\sqrt{(h'-x)}\sqrt{(2lx-x^2)}}$ generated whilst x from 0, becomes $2l$,

to the fluent of $\frac{v}{\sqrt{(h-v)}\sqrt{(2lv-v^2)}}$ generated whilst v from 0 becomes h .

From what is done at page 538, art. 460, vol. 2. of Simpson's Fluxions, it appears that the fluent of $\frac{x}{\sqrt{(h'-x)}\sqrt{(2lx-x^2)}}$

when $x = 2l$, is

$$\frac{2p}{\sqrt{h'}} \times \left(1 + \frac{2l}{2 \cdot 2 \cdot h'} + \frac{3 \cdot 3 \cdot (2l)^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot h'^2} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot (2l)^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot h'^3} + \&c. \right)$$

where p denotes the length of the quadrantal arc of a circle whose radius is 1.

Also the fluent of $\frac{v}{\sqrt{(h-v)}\sqrt{(2lv-v^2)}}$ generated whilst v , from 0 becomes h is

$$\frac{2p}{\sqrt{(2l)}} \times \left(1 + \frac{h}{2 \cdot 2 \cdot 2l} + \frac{3 \cdot 3 \cdot h^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot (2l)^2} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot h^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot (2l)^3} + \&c. \right)$$

Now it is evident by mere inspection, that if we put $\frac{h}{2l} = \frac{2l}{h'}$ the two series will become identical, and therefore the ratio of the two fluents in the before mentioned circumstances

will be barely that of $\frac{2p}{\sqrt{h'}} : \frac{2p}{\sqrt{(2l)}} :: \sqrt{\frac{2l}{h}} = m : 1$.

Q. E. D.

XVIII. QUESTION 108, by Mr. JAMES CUNLIFFE.

It is required to find two square numbers, a^2 , b^2 , and a certain number n , which being either added to, or subtracted from either of the squares a^2 , b^2 , the sums and differences thence arising shall all be rational squares.

SOLUTION,

SOLUTION, by Mr. CUNLIFFE, the Proposer.

By the question we shall have

$$a^2 + n = d^2$$

$$a^2 - n = c^2$$

$$b^2 + n = f^2$$

$$b^2 - n = g^2.$$

The sum of the two first equations gives $2a^2 = d^2 + c^2$, and their difference $2n = d^2 - c^2$, or $n = \frac{d^2 - c^2}{2}$.

Whence it appears that c^2 , a^2 , and d^2 , are three squares in arithmetical progression, whose common difference is $n = \frac{d^2 - c^2}{2}$; wherefore we may put

$$c = q^2 - p^2 + 2pq, a = p^2 + q^2, \text{ and } d = p^2 - q^2 + 2pq.$$

$$\text{Whence } n = \frac{d^2 - c^2}{2} = 4pq(p^2 - q^2).$$

Again, by adding the third and fourth equations together we get $2b^2 = f^2 + g^2$; and taking their difference $2n = f^2 - g^2$, or $n = \frac{f^2 - g^2}{2}$.

Whence it appears that g^2 , b^2 , and f^2 , are also three squares in arithmetical progression whose common difference is $n = \frac{f^2 - g^2}{2}$; wherefore we may again put

$$g = s^2 - r^2 + 2rs, b = r^2 + s^2 \text{ and } f = r^2 - s^2 + 2rs.$$

$$\text{Whence } n = \frac{f^2 - g^2}{2} = 4rs(r^2 - s^2), \text{ Consequently,}$$

$$4pq(p^2 - q^2) = 4rs(r^2 - s^2), \text{ or } pq(p^2 - q^2) = rs(r^2 - s^2).$$

Take $p = r$, and the preceding expression will become

$$rq(r^2 - q^2) = rs(r^2 - s^2); \text{ from whence } r^2 = \frac{q^3 - s^3}{q - s} = q^2 + qs + s^2;$$

$$\text{put } r = \frac{ts}{v} - q, \text{ then } r^2 = \left(\frac{ts}{v} - q\right)^2 = \frac{t^2 s^2}{v^2} - \frac{2tsq}{v} + q^2 = q^2 + qs + s^2;$$

$$\text{whence } s = \frac{q(v^2 + 2tv)}{t^2 - v^2}; \text{ take } q = t^2 - v^2, \text{ then } s = v^2 + 2tv;$$

whence

whence $p = r = \frac{ts}{v} - q = t^2 + tv + v^2$,

$$a = p^2 + q^2 = (t^2 + tv + v^2)^2 + (t^2 - v^2)^2,$$

$$b = r^2 + s^2 = (t^2 + tv + v^2)^2 + (v^2 + 2tv)^2,$$

$$\text{and } n = 4pq(p^2 - q^2) = 4(t^2 - v^2)(t^2 + tv + v^2) \times \{ (t^2 + tv + v^2)^2 - (t^2 - v^2)^2 \},$$

where t and v may be expounded by any numbers at pleasure.

Example. Take $t = 2$, and $v = 1$; then

$$a = 7^2 + 3^2 = 49 + 9 = 58, \text{ or } a^2 = (58)^2 = 3364;$$

$$b = 7^2 + 5^2 = 49 + 25 = 74, \text{ or } b^2 = (74)^2 = 5476; \text{ and}$$

$$n = 12 \times 7 \times 40 = 3360, \text{ and because these values of } a^2, b^2, \text{ and } n, \text{ are all divisible by the square number } 4, \text{ we may take } a^2 = 841, b^2 = 1369, \text{ and } n = 840.$$

Solutions were also received from Messrs. Bazley and A. M.

XIX. QUESTION 109, by MARLOVIENSIS.

In Counsellor Cooper's Letters on the Slave trade, we meet with the following question, and an algebraic formula annexed, exhibiting a general solution thereto.

"It is required to shew the average annual import of negro slaves (and the whole number imported), having given the annual decrement, or proportion of deaths to the whole stock, and the number of existing stock at the end of a given term?

"General Solution.

$$\frac{ba^n}{(a-1) \times (a^n - (a-1)^n)} = \text{the average annual import.}$$

"Where n denotes the given interval in years.

b , the stock existing at the end of that term.

a , the reciprocal of the annual decrement.

"Example. The first export of negro slaves from Africa was in the year 1503, and in 1770, being an interval of 276 years, it was found, from a pretty exact estimate, that the number of negro slaves then existing in the European, &c. powers, was five millions: the proportion of deaths among them was also determined from a series of observations to be about 1 in 20.

These

These numbers substituted in the above formula, give
 the average annual import 263,158
 the total imported (in 276 year) 70,263,186!

And, we may add, if estimated to the present time, we behold near 80 millions of the human species, the innocent victims of avarice, cruelty, and wretchedness, by this diabolical traffic! an investigation of the above *formula* is required?

FIRST SOLUTION, by Mr. LOWRY.

Put x for the number of slaves imported annually, $c = 1 - \frac{1}{a} = \frac{a-1}{a}$, (1 in a , or $\frac{1}{a}$, being the annual decrement);

Then

cx , will be the number remaining at the end of the first year;
 $x + cx$, the augmented stock;
 $cx + c^2x$, the number remaining at the end of the second year;
 $x + cx + c^2x$, the augmented stock;
 $cx + c^2x + c^3x$, the number remaining at the end of the third year and so on. Consequently,
 $cx + c^2x + c^3x + \dots + c^nx$, will be the number remaining at the end of n years.

Or, $cx (1 + c + c^2 + c^3 + \dots + c^{n-1}) = b$:

but the sum of the geometrical series $1 + c + c^2 + c^3 + \dots$

$$c^{n-1} \text{ is } = \frac{1 - (c)^n}{1 - c}.$$

Therefore $cx \times \left\{ \frac{1 - (c)^n}{1 - c} \right\} = b$. Hence

$$x = \frac{b(1-c)}{c \left\{ 1 - (c)^n \right\}} = \frac{ba^n}{(a-1) \times \left\{ a^n - (a-1)^n \right\}}, \text{ the given formula.}$$

SECOND SOLUTION, by Mr. BAZLEY.

This formula may be investigated by induction, as follows. Put the average annual import $= x$; then by the nature of the question, the stock existing at the end of the first year is

$$b = x$$

$$b = x - \frac{x}{a}, \text{ or } x = \frac{ba}{a-1} = \frac{ba}{(a-1) \times \{a-(a-1)\}}.$$

Now for the second year, the increased flock is

$$x - \frac{x}{a} + x = \frac{2ax - x}{a},$$

and this decreased by the deaths is

$$\frac{2ax - x}{a} - \frac{2ax - x}{a^2} = \frac{2a^2x - 3ax + x}{a^2} = b = \text{flock}$$

at the end of two years; hence we find

$$x = \frac{ba^2}{2a^2 - 3a + 1} = \frac{ba^2}{(a-1) \times \{a^2 - (a-1)^2\}}.$$

Again, for the third year, the increased flock is

$$\frac{2a^2x - 3ax + x}{a^2} + x = \frac{3a^2x - 3ax + x}{a^2},$$

and this decreased by the deaths as before, is

$$\frac{3a^2x - 3ax + x}{a^2} - \frac{3a^2x - 3ax + x}{a^3} = \frac{3a^3x + 6a^2x + 4ax - x}{a^3}.$$

$= b = \text{flock at the end of three years; and hence}$

$$x = \frac{ba^3}{3a^3 - 6a^2 + 4a - 1} = \frac{ba^3}{(a-1) \times \{a^3 - (a-1)^3\}}.$$

And thus we may proceed for any number (n) of years and shall find,

$$x = \frac{ba^n}{(a-1) \times \{a^n - (a-1)^n\}}.$$

THIRD SOLUTION, *by* MARLOVIENSIS.

Let x denote the number of annual exports, $\frac{1}{a} \left(\frac{1}{20} \right)$ the given decrement, n (276) the number of years in the given interval, and b (5000000) the existing flock at the end of the given term; then, by taking the decrement from the first annual export, and continually

continually adding the same export, and deducting the decrement of each successive year's stock as per quest. we have (by reducing the arising terms,) the existing stock at the end of the 1st. 2d. 3d. 4th. &c. years respectively expressed by

$$\frac{(a-1)x}{a}, (2a-1) \times \frac{(a-1)x}{a^2}, (3a^2-3a+1) \times \frac{(a-1)x}{a^3},$$

$$(4a^3-6a^2+4a-1) \times \frac{(a-1)x}{a^4}, \&c. \text{ and therefore gene-}$$

$$\text{rally for } n \text{ years, by } na^{n-1} - \frac{n \cdot n-1}{1 \cdot 2} a^{n-2} + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}$$

$$a^{n-3} \text{ (till it terminates) } \times \text{ into } \frac{(a-1)x}{a^n}.$$

But this series, continued to any assigned value of n , appears (by Clarke's Sum. of Series, p. 157,) to be equal to $a^n - (a-1)^n$; hence, we have

$$\left(a^n - (a-1)^n\right) \times \frac{(a-1)x}{a^n} = b, \text{ from whence}$$

$$x = \frac{ba^n}{(a-1) \times \left(a^n - (a-1)^n\right)}; \text{ or, which is easier for com-}$$

putation, (n being here a large number,)

$$x = \frac{b}{a-1} \times \frac{1}{1 - \left(\frac{a-1}{a}\right)^n} =$$

(by substituting the given numbers) 263158, the medium annual export; and therefore 70263186 the total number of Africans sold for slaves to the year 1770.

XX. QUESTION 110, by Mr. WM. SMITH, *Liverpool*.

If a plane triangle be placed, so that two of its angular points may always be on two straight lines given by position; it is required to determine the locus of the remaining angle?

FIRST SOLUTION, *by Mr. CUNLIFFE.*

CASE 1. When the angular point is turned from the intersection of the two right lines given by position.

Let ACB (fig. 179. pl. 9.) be the given triangle, DA , DB the two right lines given by position; draw AE to meet BC in E , and make the angle AEB equal to the supplement of the angle ADB , and join DE .

Then because the angles ADB , AEB are together equal to two right angles, a circle will pass through the points A , E , B , D , and therefore the angle BDE is equal to the angle BAE , that is equal to a given angle; therefore the right line DE is given by position, and the point E will always be found in the right line DE .

Now BE is a right line of a given length intercepted between two right lines DB , DE , given by position, and C is another given point in the right line BE ; therefore the locus of C is a given ellipsis, whose centre is D , as appears from Prop. 13, Book 2d. of De L'Hospital's Conic Sections.

COR. 1. When the angle ACB is equal to the supplement of the angle ADB , the locus of C will be a right line.

COR. 2. The locus of C will be a circle when BDE is a right angle, and C is the middle of BE ; this is obvious from a known genesis of the circle.

CASE 2. When the angular point is turned towards the intersection of the two right lines given by position.

Produce BC , (fig. 180, pl. 9.) till the angle BEA be equal to the angle BDA , and draw the right line DE , then a circle will evidently pass through the points A , E , D , B ; and therefore the angle EDA is equal to the angle EBA , that is equal to a given angle; therefore DE is given by position, and BE is given in length. And for the very same reasons as before, the locus of C will be a given ellipsis, having D for its centre.

COR. 1. When the angle BCA is equal to the angle BDA , the locus of C will be a right line passing through D .

COR. 2. When BDE is a right angle, and C bisects BE , the locus of C will be a circle whose centre is D , and diameter equal to BE , as appears from a known genesis of the circle.

SECOND SOLUTION, *by Mr. JOHNSON.*

Let DX, DY , be the straight lines given by position, and ABC the given triangle, having its angular points A, B , placed on the given straight lines.

Draw DE to make the angle ADE equal to the given angle ABC , meeting BC produced if necessary in E , and join AE . Then because the angle ADE is equal to the angle ABE , the points A, D, B, E , are in the circumference of a circle, and consequently the angle AEB is equal to the angle ADB in fig. 180, or equal to its supplement in fig. 179. In either case AE is drawn to make a given angle with BC , and AC and the angle ACE are given in magnitude, therefore CE is also given in magnitude. It is therefore obvious that the straight line DE will always pass through the same given point E in the side BC , or in that side produced; or which is the same, the locus of the given point E is the straight line DE . But DE is given in position because it makes a given angle with DX , which is given in position; and DY is given in position, and BC, CE , are given in magnitude; therefore the locus of the point C is an ellipse by the latter part of Mr. Ivory's Theorem (quest. 88).

When the points C, E , coincide, it is evident that the locus of the point C is the straight line DE ; and when the angle BDE is a right angle, and $BC = CE$, the locus of C will be a circle; for in this case DC will be equal to $BC = CE =$ to a given line.

XXI. QUESTION 111, *by Porus.*

Required to inscribe, geometrically, the maximum ellipse in a given trapezium?

SOLUTION, *by Mr. Lowry.*

Let $ABCD$ (fig. 181, pl. 9.) be the given trapezium, O the centre, and POI a diameter of the required ellipse, drawn from P , the point where the ellipse touches the side AB . Produce the opposite sides of the trapezium till they meet in G and H , and through I , the extremity of the diameter PI , draw the tangent FIL meeting BC in F , CD in E , and AH in L ; then FL will be parallel to AB .

Let OK be the semi-conjugate to PI , and draw VON perpendicular to AB ; then the rectangle $ON \times OK$ is equal to the rectangle of the semi-axes, (Prop. xxi. book iii. Robertson's Conics,) and consequently, when the ellipse is a maximum, the rectangle $ON \times OK$ will be a maximum; or, since OI is equal to OP , ON will be equal to OV and VN to $2ON$; and, because the angle FBA is given, it is obvious that BF has to VN (Fn) or $2ON$ a given ratio; therefore when the ellipse is a maximum the rectangle $BF \times OK$ will be a maximum.

Now the rectangles $BP \times FI$, $AP \times IL$, $PG \times IE$ are equal, each being equal to the square of the semi-conjugate CK (Robertson's Conics, p. ix. b. ii.); therefore

$PA : BP :: FI : IL$, and by composition

$AB : BP :: FL : IL$; but by similar triangles

$BH : AB :: FH : FL$; therefore

$BH : BP :: FH : IL :: FH \times AP : IL \times AP$.

In a similar way it is shewn that

$BP : BC :: IE \times PG : CF \times PG$:

but $IL \times AP = IE \times PG$; therefore

$BH : BC :: FH \times AP : CF \times PG$, or

$BH \times AP : BC \times AP :: FH \times AP : CF \times PG$:

but $CF \times PG = CF \times AP + CF \times AG$;

therefore, by permutation and conversion

$BH \times AP : BF \times AP :: BC \times AP : BF \times AP - CF \times AG$;

and again, by permutation and conversion

$BH : CH :: BF \times AP : CF \times AG$.

Find the point Q in AG produced such, that $CH : BH :: AG : AQ$, then AQ will be given in magnitude, and the rectangle $BF \times AP$ will be equal to the rectangle $AQ \times CF$.

Let each of these rectangles be taken from the rectangle $BF \times AB$; then the rectangle $BF \times BP$ will be equal to the difference of the rectangles $BF \times AB$, $CF \times AQ$, that is equal to the difference of the rectangles $AB \times BC$, $BQ \times CF$, (because $BC - BF$ is equal to CF).

On BC take RC such, that the rectangle $RC \times BQ$ may be equal to the rectangle $AB \times BC$; then R will be a given point, and the difference of the rectangles $AB \times BC$, $BQ \times CF$, will be equal to the rectangle $RC \times RF$.

Therefore, the rectangle $BF \times BP$ is equal to the rectangle $RC \times RF$

or, $BP : RC :: RF : BF$;

but the rectangles $BP \times FI$, $AP \times IL$, being equal, FI will be to AP as IL to BP ;

And by composition, and similar triangles, as in the first part of the analysis, we have

FI

FI : FH :: AP : BH, or
 FI : FH :: AP × BF = AQ × CF : BH × BF;

but it is shewn above that

BP : RC :: RF : BF;

therefore, by compounding these two proportions, we have

FI × BP = (OK)² : FH × RC :: AQ × CF × RF : BH × (BF)².

Now, since RC, AQ, and BH are given magnitudes, it is evident that the rectangle OK × BF will be a maximum when the solid FH × CF × RF is a maximum; or if FS be drawn perpendicular to HF to meet a semicircle described on HR in S, then FH × RF will be equal to (FS)², and the solid CF × (FS)² a maximum.

Let SX be a tangent to the circle at S, and draw ST to the centre T; then when CF × (FS)² is a maximum, the subtangent XT is equal to 2FC (see Simpson on the Maxima et Minima):

But 2FC = 2FT — 2CT; therefore XT = 3FT — 2CT; and
 XT × TF = (3FT — 2CT) × FT = (ST)², Euc. vi. 8 Cor.

Therefore TF may be found (Euc. vi. 29), and then the position of FL becomes known, and the ellipse may be described by prop. xxvii. book iv. Robertson's Conics.

Or, since BF and CF are known, and the rectangle BF × AP equal to the rectangle AQ × CF; AP may be found and also BP, FL, and IL; then the semi-conjugate diameters OI, OK, will be given in magnitude and position to describe the ellipse, and the method of doing this is well known.

When the trapezium becomes a trapezoid the analysis given above fails; and in this case if BFLA be the given trapezoid having the sides FL, AB, parallel; then BF being given the ellipse will be a maximum when the semi-diameter OK is a maximum, or when the rectangle AP × IL is a maximum, or (since BP has to IL a given ratio) when the rectangle AP × BP is a maximum, which is well known to be the case when AB is bisected in P, and then FL will be bisected in I, and the diameter PI will pass through the point of intersection I.

When BFLA is a parallelogram, it is evident that the sides will be bisected in the points of contact.

XXII. QUESTION 112, by PORUS.

If one end of a string be fixed to the extremity of a body, placed on a perfectly polished plane, and the other to a point in the plane, and any force be impressed upon the body, it is required to determine its motion?

SOLUTION,

SOLUTION, by Mr. LOWRY.

Fig. 182, pl. 9. Let q be the angle which the given plane makes with the horizontal plane, C the point to which the end of the string is fastened, and P the place of the body when it is at rest, or is at the lowest point of the plane to which the string will suffer it to descend. Then, whatever force is impressed on the body, the curve described thereby must necessarily be a circle (or part of a circle) whose centre is C , and radius CP the length of the string; and since the body is acted upon by a constant accelerative force, *viz.* that of gravity diminished in the proportion of radius to the sine of q , the motion of the body in the circle on the inclined plane will be analogous to that of a body moving in a circle placed on a vertical plane. This being premised, let the force impressed on the body at P be considered as acting perpendicularly to CP , since it may always be reduced to a force acting in that direction, and let V be the velocity communicated to the body at P by the impressed force. On the diameter Pp (produced if necessary) take DP equal to the distance which a body must descend on the plane to acquire the velocity V at the point P ; then, if D be within the circle, let the chord FDG be drawn perpendicular to CP , and then F or G will be the point in the circle to which the body will ascend before its velocity is entirely destroyed, and the body will continue to oscillate in the arc FPG : but when D coincides with p , or the velocity acquired in descending through the diameter Dp is equal to V , the body will ascend to p , and will continue to ascend and descend in the semicircles PFp , PGp ; and, lastly, when D is in the diameter produced, the body will continue to revolve in the circle, and have the same velocity at p as would be acquired in descending through Dp .

To find the velocity (v) of the body at any point H in the circle, and the time (t) of describing the arc FH , let HI be drawn perpendicular to CP , and put $g = 32\frac{1}{2}$ the force of gravity, $r = CP$ the length of the string, $a = DP$, $z =$ the arch FH and $x = PI$; then, since the velocity acquired in descending through the arch FH is equal to that acquired in descending through DI , we have $v = \sqrt{(2g \cdot \sin. q \cdot DI)} = \sqrt{\{2g \cdot \sin. q \cdot (a - x)\}}$, and therefore

$$\dot{t} = \frac{-\dot{z}}{v} = \frac{-\dot{z}}{\sqrt{\{2g \cdot \sin. q \cdot (a - x)\}}} ; \text{ or, since } \dot{z} = \frac{rx}{\sqrt{(2rx - x^2)}},$$

$\dot{t}$ is =

t is = $\frac{-x}{\sqrt{\{2g \cdot \sin. q. (a-x) (2rx-x^2)\}}}$; therefore,

$$t = \frac{1}{2} \sqrt{\frac{r}{g \cdot \sin. q}} \times \int \left(\frac{-x}{\sqrt{(ax-x^2)}} \times \frac{1}{\sqrt{(1-\frac{x}{2r})}} \right) \\ = \frac{1}{2} \sqrt{\frac{r}{g \cdot \sin. q}} \times \int \left\{ \frac{-x}{\sqrt{(ax-x^2)}} \times \left(1 + \frac{1}{2} \cdot \frac{x}{2r} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^2}{4r^2} + \&c. \right) \right\}$$

and the fluent from $x = 0$ to $x = a$ gives

$$t = \frac{1}{2} \sqrt{\frac{r}{g \cdot \sin. q}} \times \left\{ 1 + \frac{1^2}{2^2} \cdot \frac{a}{2r} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{a^2}{2r^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{a^3}{2r^3} + \&c. \right\}$$

the time of a semi-oscillation when D is within the circle, p being the length of a semi-circle whose radius is 1,

Also the fluent from $x = 0$ to $x = 2r$ gives

$$t = \frac{pr}{2} \sqrt{\frac{2}{a'g \sin. q}} \cdot \left\{ 1 + \frac{1^2}{2^2} \cdot \frac{2r}{a'} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{(2r)^2}{a'^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{(2r)^3}{a'^3} + \&c. \right\}$$

the time of a semi-revolution when D is without the circle, a' being equal to PD.

And when $a (a') = 2r$, we have

$$t = \frac{p}{2} \sqrt{\frac{r}{g \cdot \sin. q}} \times \left\{ 1 + \left(\frac{1}{2} \right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 + \&c. \right\}$$

the time of ascending through the semi-circle PFp.

If the force be impressed on the body at any given point H in the circle instead of at P, it is evident that a given velocity will be communicated to the body at H, and it will move either in the circle towards F, or in the direction Hk of the impressed force till it comes to the circle at k, when the velocity will be the same as at H. In either case the points F, f, to which the body will ascend may be easily determined, and the circumstances of the motion will then be precisely the same as before.

XXIII. QUESTION 113, by SCOTICUS.

Let EF, a straight line given in magnitude, be interposed between AC, BC, two straight lines given in position; let G be a given point in that line, then it is known that the locus of G is a given ellipse. Draw ED, FD, perpendicular to the lines given by position meeting in D, and join DG, then DG shall be perpendicular to a tangent to the ellipse at G. Required the demonstration?

FIRST

FIRST SOLUTION, *by Mr. LOWAY.*

I shall first consider the case in which the straight lines AC, CB, are at right angles to each other, as this will assist us in demonstrating the proposition when the lines are placed in any other position.

Let the ellipse which is the locus of the point G (fig. 183 and 184. pl. 10.) meet AC in A and CB in B; then AC will be equal to FG, and CB to EG, that is equal to the transverse and conjugate axes. Let GD meet CA in H, and CB in I, and draw GP perpendicular to AC. Then by similar triangles AC (FG) : CB (EG) :: GD : GH, and

AC (FG) : CB (EG) :: GI : GD; therefore

AC² : CB² :: GI : GH :: CP : HP; and it follows from cor. 1, prop. xi. Emerson's Conics, that HG is perpendicular to the tangent at G.

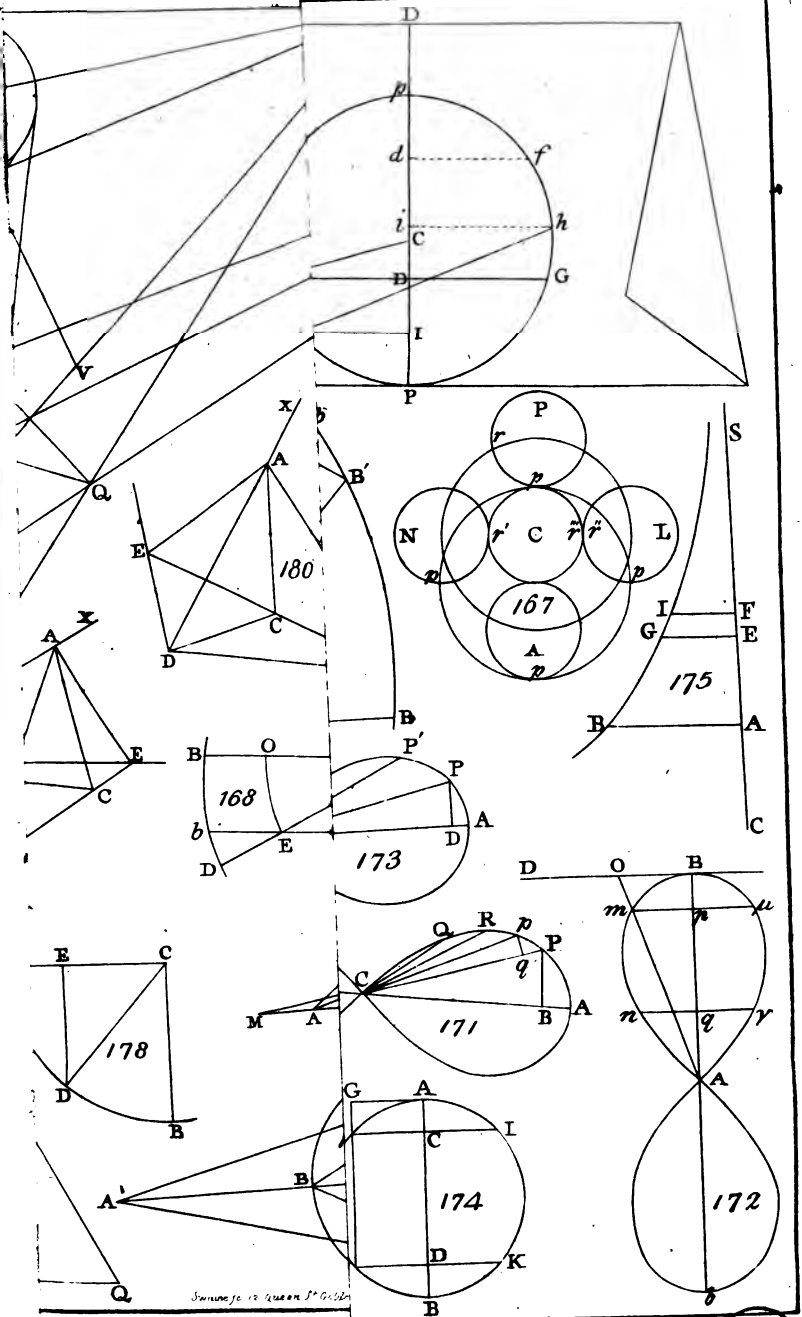
Now let the lines be placed in any other position (fig. 185, pl. 10.), and through the points F, C, E describe a circle, and let the diameter OT be drawn through the point G; and draw CO, CT, to meet the ellipse in I and K; then it is evident, from the latter part of my solution to the 88th question, that CI (=TG) and CK (=CG) will be the semi-transverse and semi-conjugate axes of the curve. Join CD, DO, DF; then because the angles CFD, CED, are right angles, CD is a diameter of the circle, and consequently DO is perpendicular to CD, and DT perpendicular to CT. Therefore between the straight lines CO, CT, which are at right angles to each other, TO is interposed so that TG, GO, are respectively equal to the semi-axes CI, CK; and OD, TD, are drawn perpendicular to the lines CO, CT; consequently DG is perpendicular to the tangent at G by the first part of the demonstration above.

When the angle FCE is a right angle, and EF is bisected in G, the ellipse becomes a circle whose radius is GC = FG = GE; and in this case the truth of the proposition is evident from the nature of the circle.

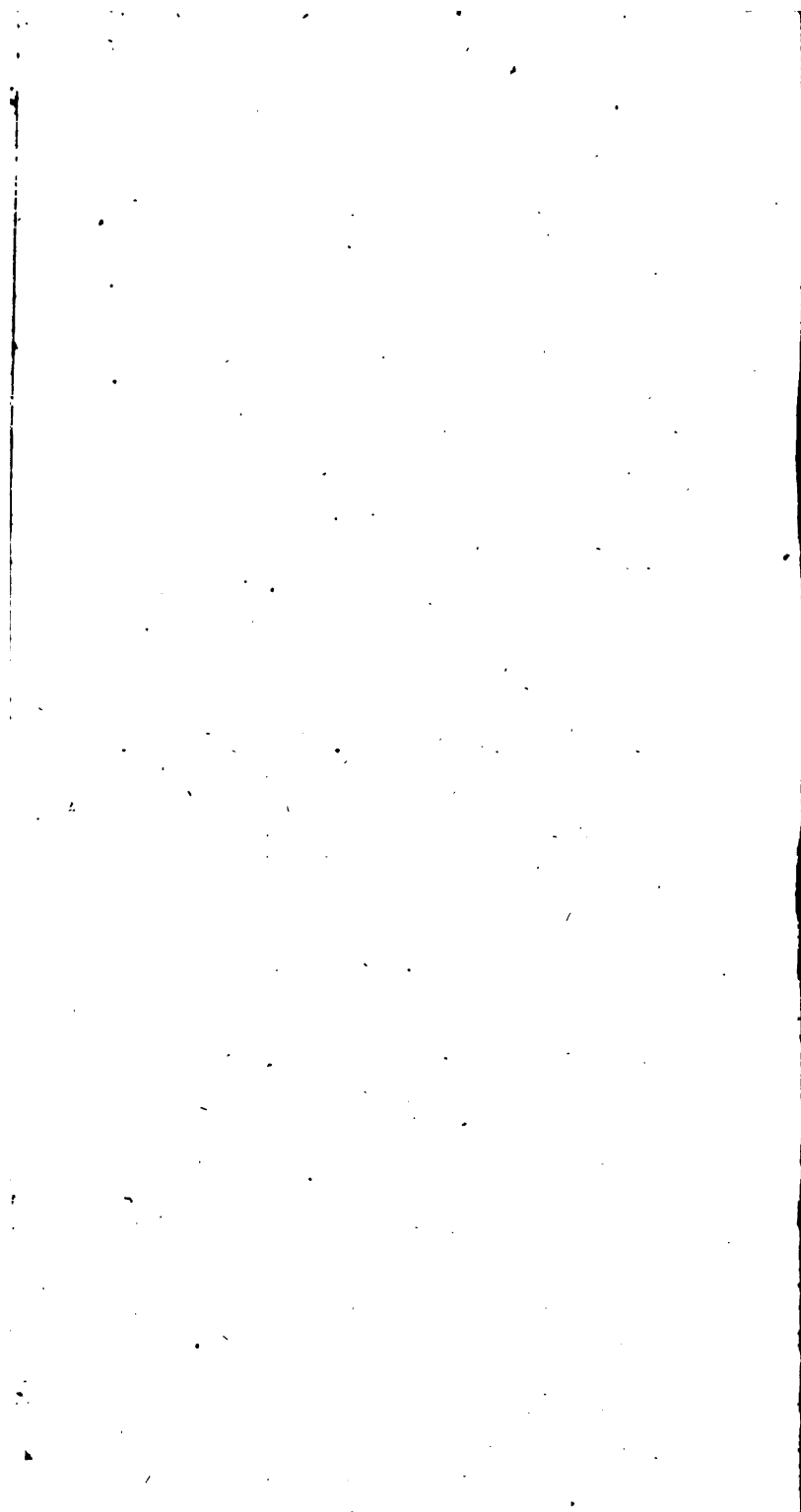
The properties of the ellipse which have been demonstrated in this and the 88th question may be elegantly applied to the solution of some useful statical problems. We select the following one as an example.

To place a given beam or rod loaded with a weight, so as to rest in equilibrio when sustained by two inclined planes given by position.

Let AC, CB, (fig. 186, pl. 10.) be the two inclined planes, making



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making given angles with the horizontal plane XCY , EF the beam or rod resting in equilibrio, and G the centre of gravity, or the place in which the weight is supposed to be concentrated. Then EG , FG , will be given in magnitude, and by the 88th question the locus of the centre of gravity G , will be an ellipse whose centre is C . Let this ellipse be described; then (by a very obvious principle of statics,) the beam or rod can only rest in equilibrio when the centre of gravity is either at the highest or lowest point to which it can possibly get; and it is evident in the present case that the centre of gravity must be at the highest point when an equilibrium takes place, for otherwise the beam or rod would not be at rest, but its centre of gravity would continue to descend till it came to one of the inclined planes; and then the beam would lie wholly on that plane.

The point G , must therefore be that point in the ellipse which is at the greatest distance from the horizontal line XCY , or the point where a tangent parallel to that line touches the curve.

Draw ED , FD , perpendicular to the inclined planes AC , CB , and let DG be produced to meet XY in P . Then, by the proposition which has been demonstrated, DG will be perpendicular to the tangent at G , or DGP will be perpendicular to the horizontal line XCY to which the tangent is parallel; it is therefore manifest that if ED , FD , be drawn perpendicular to the inclined planes at the extremities of the beam or rod, and a line DP be drawn from the point of intersection, perpendicular to the horizontal plane, it will pass through the centre of gravity of the beam or rod when it rests in equilibrio.

This conclusion agrees with prop. 53, of Emerson's Mechanics, and may serve as a confirmation of the truth of that proposition deduced from the most unexceptionable principles.

Let KDL be drawn parallel to YX meeting the planes in K and L ; then because DG is perpendicular to KL and YX , the angle FDG is the complement of the angle FDK , or equal to the angle FKD , that is equal to the given angle FCY ; and in the same manner it appears that the angle EDG is equal to the given angle ECX ; therefore in the triangle FDE , the parts FG , EG , and the angles at the vertex made by DG are given. From whence the other parts of the triangle are readily found, and the position of the beam easily determined.

Mr. Samuel Clark has considered this problem at page 37 of his Introduction to the Theory and Practice of Mechanics, where he asserts that Mr. Emerson's proposition cannot possibly be true, unless in some very particular cases there specified.

Mr. Clark has, however, been too hasty in censuring Emerson, and it may not be improper to shew, that the result in the two solutions is precisely the same, when Clark's investigation of the problem is completed.

Let s = the sine of the angle FCY,
 a = the sine of the angle ECX,
 m = the sine } of the angle ECF,
 n = the cofine }
 p = FG, and q = EG.

Then in the case of an equilibrium, Clark deduces this analogy.

FC : EC :: $qs + pna : pam^2 + nqs + pn^2a$; hence

$$\frac{EC}{FC} = \frac{\sin. CFE}{\sin. CEF} = \frac{pam^2 + nqs + pn^2a}{qs + pna} = \frac{nqs + pa}{qs + pna};$$

Therefore, by question 74 of the Repository, and reduction,

$$\frac{\cos. CFE}{\cos. CEF} = \frac{\sin. DFE}{\sin. DEF} = \frac{DE}{DF} = \frac{qs - qsn^2}{ap - pan^2} = \frac{qsm^2}{pam^2} = \frac{qs}{pa};$$

$$\text{But } \frac{DE}{DF} = \frac{\sin. FDG \times q}{\sin. EDG \times p};$$

$$\text{therefore } \frac{qs}{pa} = \frac{\sin. FDG \times q}{\sin. EDG \times p};$$

$$\text{Or, } \frac{s}{a} = \frac{\sin. FDG}{\sin. EDG}.$$

Now the sum of the angles FDG, EDG is equal to the sum of the angles whose sines are s and a , and, consequently, the angle FDG is equal to the angle whose sine is s , or equal to the angle FCY; and the angle EDG is equal to the angle whose sine is a , or equal to the angle ECX; or if KDL be drawn parallel to YX, then the angle FDG is equal to the angle FDK, and the sum of the angles FDK, FKG is a right angle; therefore the sum of the angles FDK, FDG, or the angle CDK is a right angle, and consequently DG is perpendicular to YCX.

The diameter Gg being drawn, the point g will be the lowest point in the ellipse; and would be the place of the centre of gravity of the beam or rod when it was in equilibrio if its ends were constrained to move in the lines cb , ck ; and the method of calculation in this case would be exactly the same as the preceding one.

SECOND SOLUTION, by Mr. CUNLIFFE, R. M. College.

Fig. 187, pl. 16. Join CD, and draw CQ perpendicular to DG, and upon CQ take Cm = DG. The angles CQD, CED, and CFD being all right angles, a circle whose diameter is CD will pass through the points C, Q, E, D, F: moreover the

the diameter CD is given, because FE and the angle FCE are both given.

Bisect CD in O, and join OF, OG, and OE, then the isosceles triangle FOE is entirely given, as also the line OG.

By Prop. 28, Book 2d, Emerson's Geometry,
 $(CG)^2 + (DG)^2 = (CG^2)^2 + (CM)^2 = 2(OC)^2 + 2(OG)^2$
 = a given magnitude, because CO, and OG are both given.

Again, since the points Q, E, D, F, C are in the circumference of the same circle.

$DG \times GQ = Cm \times GQ = FG \times GE =$ a given magnitude, because FG and GE are both given.

Now we have found

$(CG)^2 + (Cm)^2 =$ a given magnitude, and

$Cm \times GQ =$ a given magnitude,

which are known properties of an ellipse, whose centre is C, and one pair of conjugate semi-diameters CG and Cm; see Props. 34 and 37, Book 1st. of Emerson's Conics.

Hence, Cm, being the semi-diameter conjugate to CG, is parallel to a tangent at G, as is very well known; consequently DG is perpendicular to a tangent to the ellipse at G. *Q. E. D.*

The lengths of the principal semi-axes are $CO + OG$, and $CO - OG$; this is evident enough from what has been deduced, and the generation of the curve.

It is furthermore pretty obvious, that the sum of the two principal axes of any ellipse is less than the sum of any other pair of conjugate diameters.

For CD is the sum of the principal semi-axes which is less than $CG + DG$, the sum of another pair of conjugate semi-diameters, because any side of a plane triangle is less than the sum of the other two.

The length and inclination of that pair of conjugate diameters whose sum is a maximum, may be readily determined. In that case OG will be perpendicular to CD, and consequently CG and DG equal to each other.

The angle of inclination of any two conjugate diameters, is the complement of QGC, which in the case last-mentioned becomes known.

XXIV. QUESTION 114, by Mr. IVORY, R. M. College.

Let AB, CD, be two straight lines, cutting one another in G, and terminating in an ellipse, of which F is one focus, and draw AF, FB, FC, FD; then will

$AF \times BG + BF \times AG : DG \times FC + FD \times GC :: AB : CD.$

Required the demonstration?

G G 2

SOLUTION,

SOLUTION, *by Mr. LOWRY.*

This property is general for all the conic sections, and may be demonstrated as follows.

Fig. 188. pl. 10. Let AI, BN, CM, DL and GK, be drawn perpendicular to the directrix of the section, and draw NG to meet IA produced if necessary in V. Then, by the general property of the sections,

$AF : BF :: AI : BN$, and by similar triangles

$GB : AG :: BN : AV$, (*); therefore

$AF \times GB : BF \times AG :: AI : AV$; and by composition

$AF.GB + BF.AG : BF.AG :: IV : AV :: IV.GB : AV.GB$.

But by similar triangles,

$IV : GK :: VN : GN :: AB : GB$, or

$IV.GB = KG.AB$, and $AV.GB = BN.AG$ (*);

therefore, the proportion above becomes,

$AF.GB + BF.AG : KG.AB :: BF.AG : BN.AG :: BF : BN$.

And, in the same manner, it is shewn that,

$FC.DG + DF.CG : KG.CD :: AF : AI$, or $BF : BN$, (which is the same.) Therefore

$AF.GB + BF.AG : FC.DG + DF.CG :: KG.AB : KG.CD :: AB : CD$.

Mr. Bazley also answered this Question.

XXV. QUESTION 115, *by Mr. WALLACE, R. M. College.*

Required the locus of the angles of a parallelogram, formed by drawing tangents at the vertices of any two conjugate diameters of a given ellipse?

FIRST SOLUTION, *by HYPATIA.*

As an ellipse may be conceived to be the orthographic projection of a circle, a parallelogram formed by tangents at the vertices of any two conjugate diameters of the ellipse will be the orthographic projection of a square described about the circle, and a straight line drawn from the centre of the ellipse to any angle of the parallelogram will be the orthographical projection of a straight line drawn from the centre of the circle to the corresponding angle
of

of the square, and the former of these lines will have to the segment of it intercepted between the centre and the ellipse the same ratio that the latter has to the radius of the circle, which ratio being given, namely that of $\sqrt{2}$ to 1, the locus of the angle of the parallelogram circumscribing the ellipse will evidently be a similar concentric ellipse, having its axes to those of the other ellipse in the ratio of $\sqrt{2}$ to 1.

SECOND SOLUTION, by Mr. LOWRY.

Let ABCD (fig. 189. pl. 10.) be a parallelogram formed by tangents, drawn at the extremities of any two conjugate diameters EG, FH, and draw the diagonals AC, DB, which bisect each other in O the centre of the ellipse, as is too evident to need any demonstration is this place. Let AC meet the ellipse in *a* and *c*, and let BD meet the ellipse in *b* and *d*, and draw FG meeting OB in V; then because AF = FB and CG = BG, FG will be parallel to AC, and OV = VB = $\frac{1}{2}$ OB; also FG is ordinately applied to the diameter Ob. And, by Prop. vii. Book 2d, Robertson's Conics

OV ($\frac{1}{2}$ OB) : Ob :: Ob : OB; therefore

Ob² = $\frac{1}{2}$ OB², or 2Ob² = OB²: wherefore

OB : Ob (:: OD : Od) :: $\sqrt{2}$: 1.

In the very same way it is shewn, that

AO : Oa (:: OC : Oc) :: $\sqrt{2}$: 1.

Therefore every semi-diameter in the required curve, has a constant ratio to the corresponding semi-diameter in the ellipse; and consequently the *locus* required is an ellipse similar and concentric to the given one.

Mr. Bazley likewise answered this Question.

XXVI. QUESTION 116, by MARLOVIENSIS.

Suppose the river Thames, opposite Canvey Island, situated due east and west, to be two miles broad at high-water, and one at low, and that the tide ebbs, at certain times, six hours, the breadth decreasing from each side equal spaces in equal times. Suppose also the velocity of the water in the middle of the stream, to increase as the time for the first three hours, when it then moves six knots *per* hour; and that the velocity in the middle is to that in

in any other part, as the semi-breadth of the water, to the nearest distance of that part to the adjacent sides. These granted, what course, and how many knots *per* hour, must a boat sail from the Kentish shore directly opposite *Bowley*, on the Essex side, at high water, to make *Scar House*, on *Canvey Island*, after a three hours trip, which is exactly three miles to the eastward of the place sailed from ?

FIRST SOLUTION, by J. R. of *Manchester*.

By two approximating processes, which immediately arise from the nature of the question, it may be found, 1st, that when the vessel arrives in the middle of the stream, which is in 1 hour 5-7ths, the tide will then have set her 1·88246 miles down the stream, which is due east: and, 2^{dly}, from that time till she comes to ground on the opposite shore, which is in 1 hour 2-7ths, it will have set her 2·89039 miles more to the east. The sum of these is 4·77285, the whole distance the tide has set her east. If from this be taken 3 miles, the remainder is 1·77285, the westing she must make to gain the proposed place. There are therefore given this, her westing, and also her northing, 1·75 miles (the breadth from high water to half ebb) to find her course and distance. The first is north 45 deg. 22 min. westerly, and the other 2·48998, which divided by 3, her time of sailing, quotes ·82999 miles *per* hour, her rate of sailing.

SECOND SOLUTION, by the Proposer MARLOVIENSIS,

Let AB (fig. 100. pl. 10,) represent the breadth of the river at high water, B, *Bowley*, S, *Scar House*, P, the middle of the stream, C the place of the vessel at the end of any given time *t* from the commencement of her motion at A (which is at present supposed to be in the right line AB directly across the river), and DA the decrease of PA at the same time.

From the conditions of the question we have these analogies—

as 6 ho. : $\frac{1}{2}$ mile (decrease of PA) :: t ho. : $\frac{t}{12} = DA$; and as 3 ho. : 6 knots, or miles :: t ho. : $2t$, the velocity of the water at P.

Put PC (the distance of the boat from the middle of the stream at

at the time t) = x , then will CD, its distance from the adjacent side, be expressed by $1 - \frac{t}{12} - x$, because $PD = 1 - \frac{t}{12}$, and

$CD = PD - PC$; and, *per quest.* $1 - \frac{t}{12} : 1 - \frac{t}{12} - x :: 2t :$

$$\frac{\left(1 - \frac{t}{12} - x\right) 2t}{1 - \frac{t}{12}} = \frac{24t - 2t^2 - 24tx}{12 - t} = v,$$

a general expression for the velocity of the current in miles *per* hour, at any given time t .

Again, since the vessel is to sail at an uniform rate, and to make ground on the opposite shore at the time of half ebb, we have

AO ($1\frac{1}{2}$ miles) : 3 ho. :: AP (1 m.) : $\frac{12}{7}$ ho. (which call a), the time she comes to the middle of the stream, therefore, since $a : AP (1) :: t : \frac{t}{a} = AC$, and $PA - AC = PO$, x will be

truly expressed by $1 - \frac{t}{a} = \frac{a - t}{a}$, till she arrives at P. And

in like manner it appears that x will = $\frac{t - a}{a}$, while the vessel

sails from P to the opposite shore at O. Hence, by substituting these values of x in the above general expression for v , we have

$$\frac{24 - 2a}{a} \times \frac{t^2}{12 - t}, \text{ or, } \frac{12t^2}{12 - t} = v,$$

the velocity of the current till x from AP becomes 0; and

$$48 \cdot \frac{t}{12 - t} - 16 \cdot \frac{t^2}{12 - t} = v,$$

till x from 0 becomes PO, or .75.

Let now y denote the drift, or distance the vessel would be carried by the current from her supposed direct place at the end of the time t , and in the general fluxionary equation $\dot{y} = v\dot{t}$ (from principles well known) let the two latter values of v be written, and there will result from the former

$$\dot{y} = 12 \cdot \frac{t^2 \dot{t}}{12 - t}; \text{ the fluent of which is,}$$

while

while t from 0 becomes $\frac{12}{7}$,

$$12 \cdot (144 \text{ hyp. l. } \frac{12}{12-t} - \frac{1}{2}t^2 - 12t) =$$

in its ultimate value, $1.8826 = y$ (= PF), the distance the tide has set the vessel down the stream, or so much due east, when she arrives at the middle of the water. And from the latter value of y , arises

$j = 48 \cdot \frac{t\dot{t}}{12-t} - 16 \cdot \frac{t^2\dot{t}}{12-t}$, the fluent of which is (while t from $\frac{7}{12}$ (a) becomes 3) when corrected by considering when $y = 0$, t is = a ,

$$48 \cdot \left(12 \text{ h. l. } \frac{12}{12-t} - t - \left(12 \text{ h. l. } \frac{12}{12-a} - a \right) \right) \\ - 16 \cdot \left(144 \text{ h. l. } \frac{12}{12-t} - \frac{1}{2}t^2 - 12t - \left(144 \text{ h. l. } \frac{12}{12-a} - \frac{1}{2}a^2 - 12a \right) \right)$$

= (in its final value) $2.889 = y$ (= NK), the distance the tide has set the vessel farther east when she makes the opposite side of the river. The sum of these is $4.7716 = OK$, the whole drift easterly. But, by the quest. the vessel is to arrive at I (S), which is only three miles to the east of O (B); therefore, take $OH = (OK - OI)$ $IK = 4.7716 - 3 = 1.7716$, and it is evidently the westing she must make to counteract the tide, and be three miles to the east of the place she failed from.

Hence, $\sqrt{(AO^2 + OH^2)} = AH = \sqrt{(1.75^2 + 1.7716^2)} = 2.4901$ is the whole distance, which divided by 3, the time of sailing, gives .83 m. the uniform rate per hour; and her course ($\angle BAH$) north $45^\circ 21'$ westerly. And had the boat's destination been the same distance to the *west* instead of the *east* of the place failed from, her westing would evidently have been 1.7716 ; and thence her distance 7.966 , rate of sailing 2.655 miles per hour, and course north $77^\circ 19'$ westerly.

Rem. 1. This solution is manifestly founded on this established principle—"a body acted upon by two forces at the same time in different directions, either equably or uniformly accelerated or retarded, will pass through the same point at the end of any given time, as it would do if the two forces were each to act separately and successively for the said time"—which affords a more direct process in this case, than by investigating the nature of the curve, or vessel's real track ARI (ER being obviously = PF) by means of an equation.

2. The correction of the fluent for finding the drift from the middle

middle of the river to the opposite side, would have been unnecessary, had y and z been supposed to commence from o at F , and

therefore $x = \frac{7t}{12}$, and the final value of t been taken $= \frac{9}{7} = 1$

ho. $\frac{2}{7}$, the time of sailing from thence to K .

3. An approximate solution, independent of any fluxionary consideration, may be obtained from the above first general expression for v , by dividing the given interval (3 hours) into any number of parts, and finding the cotemporary values of t and x ; the sum of the resulting values of v will exhibit the vessel's drift from her *direct* place at the end of any assigned time t ; in which, it is evident, the greater number of divisions, or the smaller the intervals of time are, the nearer will the result approach to the truth.

XXVII. QUESTION 117, by HYPATIA.

Three circles being given by position, it is required to draw a tangent to each of them, so as to intersect in the same point, and have the segments between that point and the points of contact in a given ratio to each other?

SOLUTION, by Mr. LOWRY.

In order to simplify the solution of this problem as much as possible, we shall first find the locus of the point of intersection of tangents drawn to two circles given in magnitude and position, when these tangents have a given ratio.

Fig. 191, pl. 10. Let ID , HD , be two tangents intersecting in D ; A and B the centres, and AI , BH , radii of the given circles drawn to the points of contact. Join A , D ; B , D , and draw DV to make the angle VDI equal to the angle BDH ; then the right-angled triangles IVD , HBD , will be similar, and IV will be to BH , and VD to BD in the given ratio of ID to HD ; therefore the square of VD will have to the square of BD a given ratio: let AC be taken to BC in this ratio, and join DC ; through the point B let a circle be described to touch CD at D and meet

AD in E, and AB in F; join BE, and draw AL parallel to DC, meeting DB produced in L. Then because CD is a tangent at D, the angle BED is equal to the angle BDC (Euc. 32. iii.); that is, to the angle ALB (Euc. 29. i.); therefore the points E, B, L, A, are in the circumference of a circle: therefore

$ED \cdot DA = BD \cdot DL$: but by similar triangles

$BC : BD :: AB : BL$, and by permutation and composition,

$BC : AC :: BD : DL :: (BD)^2 : DL \cdot BD$; and, by hypothesis,

$BC : AC :: (BD)^2 : (VD)^2$; therefore

$DL \cdot DB (= ED \cdot DA) = (VD)^2$.

Now, (Euc. 47. 1.) $(VD)^2 - (IV)^2 = (ID)^2 = (AD)^2 - (IA)^2$; therefore

$(VD)^2 = (AD)^2 - \{(IA)^2 - (IV)^2\}$, and consequently

$ED \cdot DA = (AD)^2 - \{(IA)^2 - (IV)^2\}$:

let each of these be taken from $(AD)^2$, and we have

$AE \cdot AD = (IA)^2 - (IV)^2$, or $AF \cdot AB = (IA)^2 - (IV)^2$:

but IA and IV are given in magnitude, therefore the rectangle $AF \cdot AB$ is given, and consequently the point F is given. Now the points F, B, C, being given, the rectangle $BC \cdot CF$, or its equal the square of DC (Euc. 36. iii.) will be given; therefore DC will be given, and the locus of the point D is a circle whose centre is C and radius CD.

The *construction* as well as the *composition* is obvious from the analysis given above; but it is necessary to remark that when ID is less than DH, the point C must be taken on AB produced beyond A; and when ID is greater than DH, the point C must be taken on AB produced beyond B: also when ID is equal to DH, the locus is a straight line; for then DV is equal to DB,

or $(DV)^2 = (DB)^2$, that is, $(AD)^2 - \{(IA)^2 - (IV)^2\} =$

$(DB)^2$; therefore $(AD)^2 - (DB)^2 = (IA)^2 - (IV)^2$, a given magnitude, and the locus of D is a straight line perpendicular to AB, as is well known.

If the circle which is the locus of the point D, meets AB in D, and two points P and R be taken in AC such, that the rectangle $RC \cdot CP$ may be equal to the square of the radius Cd, and Pd to Rd in the given ratio of ID to HD; then if PD, RD, be drawn they will have the same ratio as the tangents ID, HD, (Playfair's Euc. prop. F. B. vi.) and P and R will be given points.

For

For by hypothesis $CP : Cd :: Cd : RC$,
 and by division $Pd : Cd :: Rd : RC$,
 and by permutation $Pd : Rd :: Cd : RC$:
 but the ratio of Pd to Rd is given, and Cd is given, therefore RC
 is given, and consequently CP is given ; and therefore the points
 P and R are given. From hence we derive the following

PORISM.

If two circles be given in magnitude and position, and tangents DI , DH , be drawn from any point to the circles so as to have a given ratio; two points P and R may be found such, that the straight lines PD , RD , being drawn from these points to the intersection of the tangents shall always have the same ratio as the tangents DI , DH .

A solution to the problem proposed is immediately deducible from what has been done above. For let A , B , O , be the centres of the given circles, and ID , HD , KD , the tangents drawn as required; then, because ID has to HD a given ratio, the locus of the point of intersection of these tangents will be a circle given in magnitude and position: also, because DH has to DK a given ratio, the locus of the point of intersection of these tangents will likewise be a circle given in magnitude and position. Consequently if these loci be described, as shewn in the analysis, their intersection will give the point required.

XXVIII. QUESTION 118, by A. B.

Let C be a body of a given weight, (suppose 875 grains), placed on a smooth and polished horizontal plane; let a thread or string perfectly unelastic and inextensible be attached to the body C , and after passing through two small rings, A and B , (so as to act on the body in a direction parallel to the horizontal plane), fall down in a vertical direction: to the extremities of the thread, let two equal weights, P and Q (suppose each = 3500 grains), be fastened; then, it is required to define the motion of the body C , and to ascertain its place at the end of any given time, supposing it to begin to move from a given place, equally distant from the two rings, abstracting from the weight of the string, friction, and the resistance of the air?

SOLUTION, by Mr. CUNLIFFE, R. M. College,

Fig. 192, pl. 10. Join AB and draw CB perpendicular to it. Then the body C, being urged by two equal forces in the directions CB, CA, by the composition of forces, will move along the diagonal of the rhombus formed by the lines CB, CA; that is, the body C will move along the perpendicular CD, and the circumstances of the motion will be precisely the same as if only half the weight C was to be drawn along the smooth inflexible rod or line CD by the descent of the weight Q.

Put $DB = a$, $BC = x$, $g = 16\frac{1}{2}$ feet, the velocity with which Q descends = v , and the contemporaneous velocity of $C = u$. Then from what is deduced in the solution to the

prize question in the Gentleman's Diary for 1788, $\frac{\frac{1}{2}Cui}{vv}$ will

express the inertia of $\frac{1}{2}C$ in the direction CB or BQ, and there-

fore $Q + \frac{\frac{1}{2}Cui}{vv}$ is the whole inertia in that direction.

Again $2gQ$ is the motive force by which the system is urged in the direction CB, therefore $2gQ \div \left(Q + \frac{\frac{1}{2}Cui}{vv} \right) = \frac{4gQvv}{2Qvv + Cui}$ expresses the accelerative force of the system in the direction CB.

Then by a known theorem $\frac{-4gQvv\dot{x}}{2Qvv + Cui} = v\dot{v}$, whence —

$4gQ\dot{x} = 2Qv\dot{v} + Cu\dot{u}$, and taking the fluents $8gQ(b-x) = 2Qv^2 + Cu^2$, where b denotes the value of BC at the commencement of the motion.

Again we shall obviously have $v^2 : u^2 :: CD^2 : CB^2$, whence $u^2 = v^2 \times \frac{CB^2}{CD^2} = \frac{v^2 x^2}{x^2 - a^2}$, and therefore $8gQ \times (b-x)$

$$= 2Qv^2 + Cu^2 = 2Qv^2 + \frac{Cv^2 x^2}{x^2 - a^2} \text{ which gives } v^2 = \frac{8gQ}{\dots}$$

$$\frac{2gQ(b-x)(x^2-a^2)}{2Q(x^2-a^2)+Cx^2} = \frac{4gm(b-x)(x^2-a^2)}{x^2-ma^2}, \text{ by put-}$$

$$\text{ting } m = \frac{2Q}{2Q+C}, \text{ whence } v = \frac{2\sqrt{g}\sqrt{(b-x)}\sqrt{(x^2-a^2)}}{\sqrt{\left(\frac{x^2}{m}-a^2\right)}}.$$

Moreover we shall have

$$\frac{-\dot{x}}{v} = -\dot{x} \sqrt{\frac{\frac{x^2}{m}-a^2}{x^2-a^2}} \div 2\sqrt{g}\sqrt{(b-x)} \text{ for the flux-}$$

ion of the time, the fluent of which is pretty obviously expressible by the time of a heavy body's descending freely, by the force of gravity, along the arch of a given hyperbola whose axis is perpendicular to the horizon,

But for the business of calculation it will be best to obtain an expression for the time by means of an infinite series,

For this purpose put $y = b - x$, or $x = b - y$, then

$$v = \frac{2\sqrt{(gm)}\sqrt{(b-x)}\sqrt{(x^2-a^2)}}{\sqrt{(x^2-ma^2)}} = 2\sqrt{(gmy)} \times \sqrt{\frac{(y-b)^2-a^2}{(y-b)^2-ma^2}}$$

$$\text{whence } \dot{t} = \frac{-\dot{x}}{v} = \frac{\dot{y}}{v} = \dot{y} \frac{\sqrt{\frac{(y-b)^2-a^2}{(y-b)^2-ma^2}}}{2\sqrt{(gm)} \times y^{\frac{1}{2}}}, \text{ the fluent of}$$

which may be readily found in an infinite series, and by reverting this series we may obtain the value of y in terms of t and given quantities.

XXIX. QUESTION 119, by PHILOMATHES OXONIENSIS.

It is required to find the fluent of $\dot{x} \sqrt{1 + \frac{1}{x^3}}$, generated, while x increases from $\frac{1}{10}$ to 10, and to give the work at large, bringing out the result true to four places of decimals, at least?

SOLUTION,

SOLUTION, by PHILOMATHES OXONIENSIS, the Proposer.

1. The given expression is evidently $= \dot{x} \sqrt{\left(\frac{1+x^3}{x^3}\right)} = \dot{x} x^{-\frac{3}{2}} \sqrt{(1+x^3)}$, the fluent of which may be assumed $= \sqrt{(1+x^3)} \times (-2x^{-\frac{1}{2}} + ax^{\frac{1}{2}} - 6x^{\frac{3}{2}} + 7x^{\frac{5}{2}} - 8x^{\frac{7}{2}}, \&c.)$; the fluxion of which expression being taken, and put $=$ the given fluxion, we shall have

$$\dot{x} x^{-\frac{3}{2}} \sqrt{(1+x^3)} = \begin{cases} \dot{x} \sqrt{(1+x^3)} \times (x^{-\frac{3}{2}} + \frac{5}{2} ax^{\frac{1}{2}} - \frac{11}{2} 6x^{\frac{3}{2}} + \frac{17}{2} 7x^{\frac{5}{2}}, \&c.) \\ + \frac{3\dot{x}x^{\frac{1}{2}}}{2\sqrt{(1+x^3)}} \times (-2x^{-\frac{1}{2}} + ax^{\frac{1}{2}} - 6x^{\frac{3}{2}} + 7x^{\frac{5}{2}}, \&c.) \end{cases}$$

which equation being multiplied by $\frac{\sqrt{(1+x^3)}}{\dot{x}}$, and the like powers of x ranged one under the other, gives

$$\frac{1+x^3}{x^{\frac{3}{2}}} = \begin{cases} (1+x^3) \times (x^{-\frac{3}{2}} + \frac{5}{2} ax^{\frac{1}{2}} - \frac{11}{2} 6x^{\frac{3}{2}} + \frac{17}{2} 7x^{\frac{5}{2}} - \frac{23}{2} 8x^{\frac{7}{2}}, \&c.) \\ - 3x^{\frac{3}{2}} + \frac{3}{2} ax^{\frac{3}{2}} - \frac{3}{2} 6x^{\frac{5}{2}} + \frac{3}{2} 7x^{\frac{7}{2}}, \&c. \end{cases}$$

And this equation, when multiplication has been actually performed in the first line of it, will become

$$x^{-\frac{3}{2}} + x^{\frac{3}{2}} = \begin{cases} x^{-\frac{3}{2}} + \frac{5}{2} ax^{\frac{1}{2}} - \frac{11}{2} 6x^{\frac{3}{2}} + \frac{17}{2} 7x^{\frac{5}{2}} - \frac{23}{2} 8x^{\frac{7}{2}}, \&c. \\ + x^{\frac{3}{2}} + \frac{5}{2} ax^{\frac{3}{2}} - \frac{11}{2} 6x^{\frac{5}{2}} + \frac{17}{2} 7x^{\frac{7}{2}}, \&c. \\ - 3x^{\frac{3}{2}} + \frac{3}{2} ax^{\frac{3}{2}} - \frac{3}{2} 6x^{\frac{5}{2}} + \frac{3}{2} 7x^{\frac{7}{2}}, \&c. \end{cases}$$

from which we have

$$5a = 6, \quad 116 = 8a, \quad 177 = 146, \quad 238 = 207, \quad \&c.$$

and $a = \frac{6}{5}, \quad 6 = \frac{8a}{11}, \quad 7 = \frac{146}{17}, \quad 8 = \frac{207}{23}, \quad \&c.$

so that the variable part of the fluent sought is

$$\sqrt{(1+x^3)} \times \left(-2x^{-\frac{1}{2}} + \frac{6}{5} x^{\frac{1}{2}} - \frac{6 \cdot 8}{5 \cdot 11} x^{\frac{3}{2}} + \frac{6 \cdot 8 \cdot 14}{5 \cdot 11 \cdot 17} x^{\frac{5}{2}} - \frac{6 \cdot 8 \cdot 14 \cdot 20}{5 \cdot 11 \cdot 17 \cdot 23} x^{\frac{7}{2}}, \&c.\right)$$

Now, although this series wants a correction, and will not converge

verge when x becomes greater than 1, still it is very useful. For, when $x = \frac{1}{10}$, this expression becomes

$$\sqrt{(1.001)} \times \left(-\frac{2\sqrt{10}}{1} + \frac{6}{5.100\sqrt{10}} - \frac{6.8}{5.11.100000\sqrt{10}}, \&c. \right);$$

when $x = 1$, this expression becomes

$$\sqrt{2} \times \left(-2 + \frac{6}{5} - \frac{6.8}{5.11} + \frac{6.8.14}{5.11.17}, \&c. \right);$$

and the difference of these two quantities, viz.

$$\sqrt{(1.001)} \times \left(\frac{2\sqrt{10}}{1} - \frac{6}{5.100\sqrt{10}} + \frac{6.8}{5.11.100000\sqrt{10}}, \&c. \right);$$

$$- \sqrt{2} \times \left(2 - \frac{6}{5} + \frac{6.8}{5.11} - \frac{6.8.14}{5.11.17}, \&c. \right),$$

is the quantity generated while x increases from $\frac{1}{10}$ to 1. The former of these series converges very swiftly, and the latter may easily be computed.

2. We now want a convenient expression of the part of the fluent generated while x increases from 1 to 10; which is easily attainable as follows:

Assume the fluent of $\dot{x}\sqrt{(1+x^{-3})} = \sqrt{(1+x^{-3})} \times (x-ax^{-2} + bx^{-5} - cx^{-8} + dx^{-11}, \&c.)$, and put the fluxion of this expression = the given fluxion, and there will be

$$\dot{x}\sqrt{(1+x^{-3})} = \begin{cases} \dot{x}\sqrt{(1+x^{-3})} \times (1+2ax^{-3} - 5bx^{-6} + 8cx^{-9} - 11dx^{-12}, \&c) \\ - \frac{3\dot{x}}{2x^4\sqrt{(1+x^{-3})}} \times (x-ax^{-2} + bx^{-5} - cx^{-8} + dx^{-11}, \&c) \end{cases}$$

and this equation multiplied by $\frac{\sqrt{(1+x^{-3})}}{\dot{x}}$, and properly arranged, becomes

$$1+x^{-3} = \begin{cases} (1+x^{-3}) \times (1+2ax^{-3} - 5bx^{-6} + 8cx^{-9} - 11dx^{-12}, \&c. \\ - \frac{3}{2}x^{-3} + \frac{3}{2}ax^{-6} - \frac{3}{2}bx^{-9} + \frac{3}{2}cx^{-12}, \&c.; \end{cases}$$

that

that is, by actual multiplication,

$$1+x^{-3} = \begin{cases} 1 + 2ax^{-3} - 5bx^{-6} + 8cx^{-9} - 11dx^{-12}, & \&c. \\ + \quad \quad \quad -3 \quad + 2ax^{-6} - 5bx^{-9} + 8cx^{-12}, & \&c. \\ - \frac{3}{2}x^{-3} + \frac{3}{2}ax^{-6} - \frac{3}{2}bx^{-9} + \frac{3}{2}cx^{-12}, & \&c. \end{cases}$$

and hence we have

$$2a = \frac{3}{2}, \quad 5b = \frac{7}{2}a, \quad 8c = \frac{13}{2}b, \quad 11d = \frac{19}{2}c, \quad \&c.$$

$$\text{and } a = \frac{3}{4}, \quad b = \frac{7}{10}a, \quad c = \frac{13}{6}b, \quad d = \frac{19}{22}c, \quad \&c.$$

We therefore now have another expression of the variable part of the fluent sought, viz :

$$\sqrt{(1+x^{-3})} \cdot (x \frac{3}{4}x^{-2} + \frac{3 \cdot 7}{4 \cdot 10}x^{-5} - \frac{3 \cdot 7 \cdot 13}{4 \cdot 10 \cdot 16}x^{-8} + \frac{3 \cdot 7 \cdot 13 \cdot 19}{4 \cdot 10 \cdot 16 \cdot 22}x^{-11}, \quad \&c.)$$

which series will converge while x increases from 1 to 10, and will serve our purpose. For,

when $x = 1$, this expression becomes

$$\sqrt{2} \times (1 - \frac{3}{4} + \frac{3 \cdot 7}{4 \cdot 10} - \frac{3 \cdot 7 \cdot 13}{4 \cdot 10 \cdot 16} + \frac{3 \cdot 7 \cdot 13 \cdot 19}{4 \cdot 10 \cdot 16 \cdot 22}, \quad \&c.);$$

when $x = 10$, this expression becomes

$$\sqrt{(1 \cdot 001)} \times (10 - \frac{3}{4 \cdot 100} + \frac{3 \cdot 7}{4 \cdot 10 \cdot 100000}, \quad \&c.);$$

and the difference of these two quantities, viz.

$$\sqrt{(1 \cdot 001)} \times (10 - \frac{3}{4 \cdot 100} + \frac{3 \cdot 7}{4 \cdot 10 \cdot 100000}, \quad \&c.$$

$$- \sqrt{2} \times (1 - \frac{3}{4} + \frac{3 \cdot 7}{4 \cdot 10} - \frac{3 \cdot 7 \cdot 13}{4 \cdot 10 \cdot 16} + \frac{3 \cdot 7 \cdot 13 \cdot 19}{4 \cdot 10 \cdot 16 \cdot 22}, \quad \&c.)$$

is the quantity generated while x increases from 1 to 10.

3. All that is now wanting to a full answer to this question is the decimal value of the numerical series obtained in the two preceding articles ; and these may be computed as follows :

Computation

Computation of the values of the two swiftly converging series.

$$\begin{array}{rcl}
 2\sqrt{10} = 6.324555, & 10 = 10.000000, \\
 + \frac{6.8}{5.11100000\sqrt{10}} = \frac{0.000003}{+ 6.324558}, & + \frac{3.7}{4.10.100000} = \frac{0.000005}{+ 10.000005}, \\
 - \frac{6}{5.100\sqrt{10}} = \frac{-0.370005}{- 6.320763}, & - \frac{3}{4.100} = \frac{-0.007500}{- 9.992500}, \\
 \text{Value of three terms} & 9.992505. & \text{Value of three terms} & 9.992505.
 \end{array}$$

Sum $16.313268 \times \sqrt{(1.001)} = 16.321425$, the value of the two swiftly converging series when multiplied by their common factor.

Computation of the values of the two slowly converging series.

The first thirteen terms of the series $a - \frac{6}{5} + \frac{6.8}{5.11}$, &c. expressed in decimals continued to six places of figures, will be $2.000000 - 1.200000 + 0.872727 - 0.718717 + 0.624971 - 0.560319 + 0.512292 - 0.474807 + 0.444500 - 0.419340 + 0.398017 - 0.379647 + 0.363606$. The sum of the first six terms is 1.018662 , and if the remaining terms be transformed into a more swiftly converging series, by the method described in *Simpson's Dissertations*, p. 62, we shall have $D' = 0.037485$, $D'' = 0.007178$, $D''' = 0.002031$, $D^{IV} = 0.000721$, $D^V = 0.000295$, and $D_{VI} = 0.000129$; and the terms of the more swiftly converging series will be

$$\begin{array}{rcl}
 \frac{1}{2} \times 0.512293 & = & 0.256146, \\
 \frac{1}{4} D' & = & 0.009371, \\
 \frac{1}{4} D'' & = & 0.000897, \\
 \frac{1}{16} D''' & = & 0.000127, \\
 \frac{1}{32} D^{IV} & = & 0.000023, \\
 \frac{1}{64} D^V & = & 0.000005, \\
 \frac{1}{128} D^{VI} & = & 0.000001,
 \end{array}$$

Value of this series 0.266570 ,
Six terms above computed 1.018662 ,

1.285232 , value of the original series.

The first thirteen terms of the series $1 - \frac{3}{4} + \frac{3 \cdot 7}{4 \cdot 10}$, &c. expressed in decimals as above, will be $1 \cdot 000000 - 0 \cdot 750000 + 0 \cdot 525000 - 0 \cdot 426563 + 0 \cdot 368395 - 0 \cdot 328924 + 0 \cdot 299901 - 0 \cdot 277409 + 0 \cdot 259317 - 0 \cdot 244356 + 0 \cdot 231717 - 0 \cdot 220855 + 0 \cdot 211390$. The sum of the first six terms is $0 \cdot 387008$; and if the remaining terms be transformed into a more swiftly converging series, by the method above mentioned, we shall have $D^I = 0 \cdot 022492$, $D^{II} = 0 \cdot 004400$, $D^{III} = 0 \cdot 001269$, $D^{IV} = 0 \cdot 000460$, $D^V = 0 \cdot 000196$, $D^{VI} = 0 \cdot 000097$; and the terms of the equivalent series will be

$$\begin{aligned} \frac{1}{2} \times 0 \cdot 299901 &= 0 \cdot 149951, \\ \frac{1}{4} D^I &= 0 \cdot 005623, \\ \frac{1}{4} D^{II} &= 0 \cdot 000550, \\ \frac{1}{16} D^{III} &= 0 \cdot 000079, \\ \frac{1}{32} D^{IV} &= 0 \cdot 000014, \\ \frac{1}{64} D^V &= 0 \cdot 000003, \\ \frac{1}{128} D^{VI} &= 0 \cdot 000001; \end{aligned}$$

Value of this series . . . $0 \cdot 156221$
Six terms above computed $0 \cdot 387908$

$0 \cdot 544129$, the value of the original series,
 $1 \cdot 285232$, the value of the other slowly
[converging series.]

Sum $1 \cdot 829361 \times \sqrt{2} = 2 \cdot 587107$, the value of the two slowly converging series when multiplied by their common factor.

We have now only to take $2 \cdot 587107$, the value of the two negative series, from $16 \cdot 321425$, the value of the two affirmative ones, and there will remain $13 \cdot 734318$, the value of the fluent required.

XXX. PRIZE QUESTION 120, by EDINBURGENSIS.

A polygon being given in magnitude and position, it is required to describe a polygon of an equal number of sides, having its several angles on the several sides of the given polygon, and such that the perimeter shall be the least possible?

SOLUTION, by Mr. LOWRY, *R. M. College.*

PROPOSITION. A LEMMA. (Fig. 193. pl. 10.)

Let $ABCD \dots E$ be a given polygon, and $PQRS \dots T$ a polygon of an equal number of sides inscribed in, or having its several angles placed on the several sides of, the given polygon : then the perimeter of the inscribed polygon will be the least possible, when each two of its adjoining sides make equal angles with the corresponding side of the circumscribing polygon ; that is, when the angle TPA is equal to the angle QPB , the angle PQB to the angle RQC , the angle QRC to the angle SRD , the angle RSD to the angle TSE , the angle STE to the angle PTA ; and so on.

For let $pQRST \dots p$ be any other polygon inscribed in the given one, and let two of its sides, as Tp , Qp make unequal angles with the side AB at p . Draw TIK perpendicular to AB , meeting QP produced in K and join KP . Then because the angle TPI is equal to the angle QPB , that is, to the angle IPK , the right angled triangles TPI , KPI will be equal in every respect, therefore PT is equal to PK and IT to IK , and consequently in the right angled triangles ITp , IKp , the sides Tp , Kp will be equal ; therefore KQ is equal to the sum of the sides TP , QP , and $Kp + pQ$ equal to the sum of the sides Tp , Qp : but KQ is less than $Kp + pQ$, and therefore the sum of the sides TP , QP is less than the sum of the sides Tp , pQ , and consequently the perimeter of the polygon $PQRST \dots P$ is less than the perimeter of the polygon $pQRST \dots p$. And the same may be proved, if any other two adjoining sides make unequal angles with the corresponding side of the circumscribing polygon ; and therefore the perimeter of the polygon $PQRST \dots P$ must be less than the perimeter of any other polygon that can be inscribed in the given one.

This being demonstrated, it remains only to determine the position of the inscribed polygon, so that each two of its adjoining sides may make equal angles with the corresponding side of the given polygon ; and to effect this in as perspicuous a manner as possible, I shall consider the problem in distinct cases, according as the polygon consists of an even or of an odd number of sides.

CASE I.

When the number of the sides of the polygon is odd. Fig. 194, pl. 10.

Let $ABCDE$, &c. be the given polygon, and $abcde$, &c. the inscribed polygon whose sides make equal angles with the corresponding sides of the given one, and let the angles of the polygon $ABCDE$, &c. be denoted by the letters placed at the angular points, viz. the angle EAB by A , the angle ABC by B , and so on. Also let the angle Aas , or Bab be denoted by a , the angle Bba , or Cbc by b , and so on; and let π be put for two right angles. Then, because the sum of the three angles in each of the triangles Bab , Cbc , &c. is equal to two right angles, we have

$$B + a + b = C + b + c = D + c + d = E + d + e = A + e + a, \&c. = \pi,$$

and taking the sum of the alternate terms

$$B + a + b + D + c + d + A + e + a = C + b + c + E + d + e + \pi,$$

$$\text{or } 2a = \pi - B + C = D + E - A, \&c.;$$

and in a similar way we find

$$2b = \pi - C + D - E + A - B, \&c.$$

$$2c = \pi - D + E - A + B - C, \&c.$$

$$2d = \pi - E + A - B + C - D, \&c.$$

$$2e = \pi - E - A + B - C + D, \&c.$$

The angles being found, we have only to determine the position of one of the angular points a of the inscribed polygon, and to do this, let the sides ab , bc , cd , &c. be produced till the distances bi , cm , dp , es , af , &c. are equal to each other, and of any assumed magnitude. And let the perpendiculars ikl , mno , pqr , stu , fgh , &c. be drawn to the sides of the given polygon, meeting the sides of the inscribed one in l , o , r , u , h , &c. Then, because of the equal angles at a , b , c , d , e , &c. it is evident that ik will be equal to kl , mn to no , pq to qr , st to tu , fg to gh ; also, bi will be equal to bl , cm to co , dp to dr , es to eu , and af to ah , &c.

Now let P be any point in AB , and let PQ , QR , RS , ST , TU , &c. be drawn parallel to the sides ab , bc , cd , de , ea , &c. respectively, and let aY be perpendicular to BC , and bZ perpendicular to AB produced. Then, because $aB : bZ = Bb : aY$, we have $aB : Bb :: aY : bZ$: but by similar triangles $ab : aY :: bi : ik$, and $ab : bz :: af (ah) : gh$; therefore $aY : bZ :: ik : gh$, or $aB : Bb :: ik : gh = fg$, or, because PQ , ab are parallel,

af

$$aP : bQ :: ik : gh = fg ;$$

and in the same way it is shewn that

$$bQ : cR :: nm : kl = ik$$

$$cR : dS :: pq : on = nm$$

$$dS : eT :: st : nq = pq, \&c \text{ to}$$

$$eT : aU :: fg : ut = st ;$$

And by compounding these proportions, it is obvious that we shall finally obtain

$$aP : aU :: fg : fg :$$

Therefore aP is equal to aU . Whence it follows that wherever the point P is taken in AB , the distance UP , will always be bisected in a ; and from hence we derive the following simple and general construction.

Assume any point P in one of the sides, as AB , and draw PQ , QR , RS , ST , TU , &c. to make the given angles with the sides AB , BC , &c. as determined in the analysis, bisect UP in a , and a will be the position of one of the angular points of the required polygon.

If $n + 1$ be the number of the sides of the polygon, it is obvious from comparing the values of the angles as determined above, that the polygon can only be inscribed in the manner specified in the proposition, when the sum of $n + 1$ of the angles is less than the sum of the n remaining angles, together with two right angles.

From the point U , let Ua' be drawn parallel to ab , $a'b'$ to bc , $b'e'$ to cd , $c'd'$ to de , $d'e'$ to ea , &c. then it is evident from above that ae' will be equal to aU , that is, equal to aP ; therefore the points P and e' coincide. From whence we may draw the following curious and remarkable inference, *viz.* that beginning at any point P in one of the sides AB , and drawing the lines PQ , QR , RS , &c. parallel to the sides of the inscribed polygon and continuing them twice round the figure, the last side de' will terminate in the point P , where we originally began.

The perimeter of the figure made by going twice round in the manner just described, will be equal to twice the perimeter of the inscribed polygon $abcde$, &c. For since aP is equal to aU , Qb will be equal to ba' , and consequently $PQ + Ua'$ will be equal to twice ba ; and in the same way it appears that $QR + a'b'$ is equal to twice bc , $RS + b'e'$ to twice cd , and so on; therefore the perimeter of the figure $PQRSTU a'b'c'd'P$ is equal to twice the perimeter of the polygon $abcde$, &c.

In the triangle ABC (fig. 195, pl. 10.) we have $2a = n - B + C - A = 2C$, or $a = C$, and in the same way we obtain $b = A$, and $c = B$; hence the angles which the sides of the inscribed triangle make with the sides of the given triangle are equal to the angles opposite those sides.

It is also obvious that each of the angles of the given triangle must be less than a right angle.

If

If the lines aC , bA , cB be drawn; they will be perpendicular to the sides AB , BC , AC , respectively, as is too evident to need demonstrating here.

CASE II.

When the given polygon consists of an even number of sides, (Fig. 196, pl. 10).

Denoting, as before, the angles of the given polygon by A , B , C , &c. and the angles made by the sides of the inscribed polygon and the sides AB , BC , CD , &c. of the given one by a , b , c , d , &c. we have.

$B + a + b = C + b + c = D + c + d$, &c., ... = to $f + a + A$, therefore

$B + a + b + D + c + d + F + e + f$ &c. = $C + b + c + E + d + e + f + a + A$, &c. or $B + D + F +$ &c. = $C + E + A +$ &c. :

which shews that the sides of the polygon $abcdef$, &c. can only make equal angles with the corresponding sides of the given one, when the sum of the angles B , D , F , &c. is equal to the sum of the angles C , E , A , &c. which will always be the case when the given polygon can either be inscribed in a circle, or is equiangular to one that can be so inscribed.

It shews also, that the angles a , b , c , &c. are not determinable from the angles of the given polygon as in the case of the odd sided figures, but that their values depend on the relations of the sides of the polygon.

To determine the angles, let perpendiculars BQ , BI be drawn from any of the angular points as B , to the sides CD and AF , and continued till LQ be equal to BL and IP to IB : also from the points Q and P , let perpendiculars QM , PW be drawn to the sides ED , EF , adjoining the sides to which the other perpendiculars were drawn, and let these perpendiculars be continued till MO be equal to QM and WS to PW ; and proceed in the same manner with the remaining sides if there be any.

Let the line SO which joins the extremities of the perpendiculars last drawn meet the sides DE , EF to which these perpendiculars are drawn in D' and E' , and draw $D'Q$ meeting DC in C' and $E'P$ meeting AF in F' and join BC' , BF' ; then the angles ABF' , CBC' , $DC'D$, $ED'E$, $FE'F$, &c. to AFB will be respectively equal to the angles a , b , c , d , e , f . That is, if the sides bc , cd , de , ef and af be drawn parallel to the lines BC' , $C'D$, $D'E$, $E'F$, FB respectively, and ab be joined, the sides of the polygon $abcdef$, &c. will make equal angles with the corresponding sides of the given polygon.

Join CQ , AP , and let de produced meet CQ in R ; also let ef produced meet AP in T . Then because BL is equal to LQ
and

and the angles at L right angles, the triangles $BC'L$, $C'LQ$ are equal in every respect, therefore the angle $c C'B$ is equal to the angle $cC'Q$, and consequently the angle bcC , or CcR equal to the angle Dcd . In the same way it is shewn that the angle Ddc is equal to the angle Ede , the angle Eed to the angle FeS , the angle Ffe to the angle Afa , &c. Again it is evident that the right-angled triangles BCL , LCQ are equal in every respect; therefore BC is equal to CQ and the angle BCL to the angle QCL ; and consequently the triangles $bc C$, $c CR$ having the angles at C and c equal, will also have the sides bC , CR equal, therefore $BC - bc$ will be equal to $CQ - CR$, or Bb will be equal to RQ ; and in the same manner it is shewn that $a B$ is equal to PT . Now let OV be drawn to make the angle QOV equal to the angle CQO , and SV to make the angle VSP equal to the angle APS , and let de be produced to meet OV in N , SV in S , and SY , drawn parallel to OV in Y . Then, the triangles $D'QM$, $D'OM$, being in every respect equal, $D'Q$ will be equal $D'O$ and the angle $D'QM'$ to the angle $D'OM$; therefore, the angle RQD' is equal to the angle NOD' , and the angles at D' and d being equal it is evident that the trapezoids $dRQD'$, $d NOD'$ are equal in every respect; therefore dR will be equal to dN and RQ or Bb to NO , that is, (because of the parallels,) to SY . It is proved in the same manner that PT or aB is equal to SX .

On OV take OZ equal to CQ , and on SV take SU equal to PA , and join DZ , FU ; then it is easily proved that the angle EDZ is equal to D , the angle DZV to $n - c$; the angle UFE to F and the angle FUV to $n - A$: also that the sum of all the angles in the figure $VZDE FUV$ is equal to $4n$; therefore

$ZVU + D + n - c + 2n - E + F + n - A = 4n$,
or $ZVU + D + F = A + C + E = B + D + F$ (by hypothesis);
therefore ZVU , or its equal the angle YSX is equal to B ; and consequently the triangles YSX , aBb , having two sides and the included angle equal, will be equal in every respect, therefore the angle $abB = SYX = dNV = dRc = Cbc$, and the angle $baB = SXY = cTA = Aaf$, therefore the sides of the inscribed polygon $abcdef$, make equal angles with the corresponding sides of the given polygon as required.

It is evident that O and S are given points, and therefore the construction for determining the angles is sufficiently obvious.

Moreover, since be is equal to CR , and dN to dR , dN is equal to $dc + cb$; and since af is equal to fT and eX to cT , eX is equal to $cf + af$; and YX has been shewn to be equal to ab ; therefore the perimeter of the inscribed polygon is equal to NG , that is, because of the parallels, to the given line OS .

If any other polygon as $a'b'c'd'e'f'$, &c. be inscribed in the given one, so that the sides $b'c'$, $c'd'$, $d'e'$, $e'f'$, $f'a'$ may be respectively parallel to the sides bc , cd , de , ef , fa ; then it is manifest from what has been done above, that the remaining sides $a'b'$, $a'b$, will also be parallel, and that the perimeter of the latter

inscribed

inscribed polygon will also be equal to OS, or equal to the perimeter of the other inscribed polygon.

Whence it appears that an indefinite number of polygons may be inscribed in the given polygon, having their sides respectively parallel and their perimeters equal to the same constant quantity, and yet less than the perimeter of any other polygon that can be inscribed in the given one, and whose sides make unequal angles with the corresponding sides of the given polygon, which is a very curious and remarkable proposition.

In the case of the rectangle it is obvious that the sides of the inscribed figure will be parallel to the diagonals, and the perimeter equal to the sum of the two diagonals.

When the sides of the inscribed polygon cannot make equal angles with the corresponding sides of the given polygon as in the cases specified in the solution, it appears that the problem is wholly indeterminate, and that the perimeter of the inscribed polygon will continue to increase or decrease till one of the sides vanishes or till two of the angular points of the inscribed polygon coincide with one of the angular points of the given polygon.

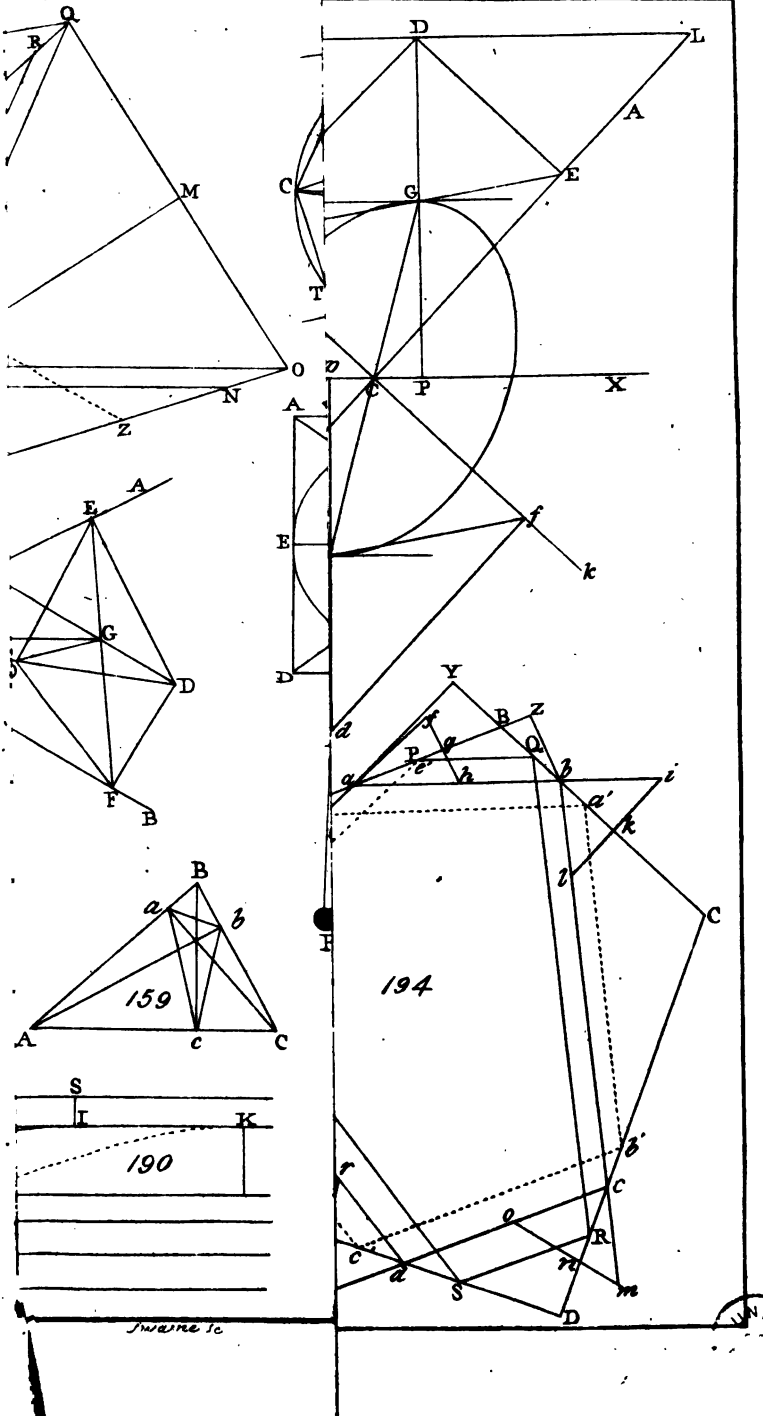
MR. LOWRY is requested to send to MR. GLENDINNING's for the Medal for solving the Prize Question.

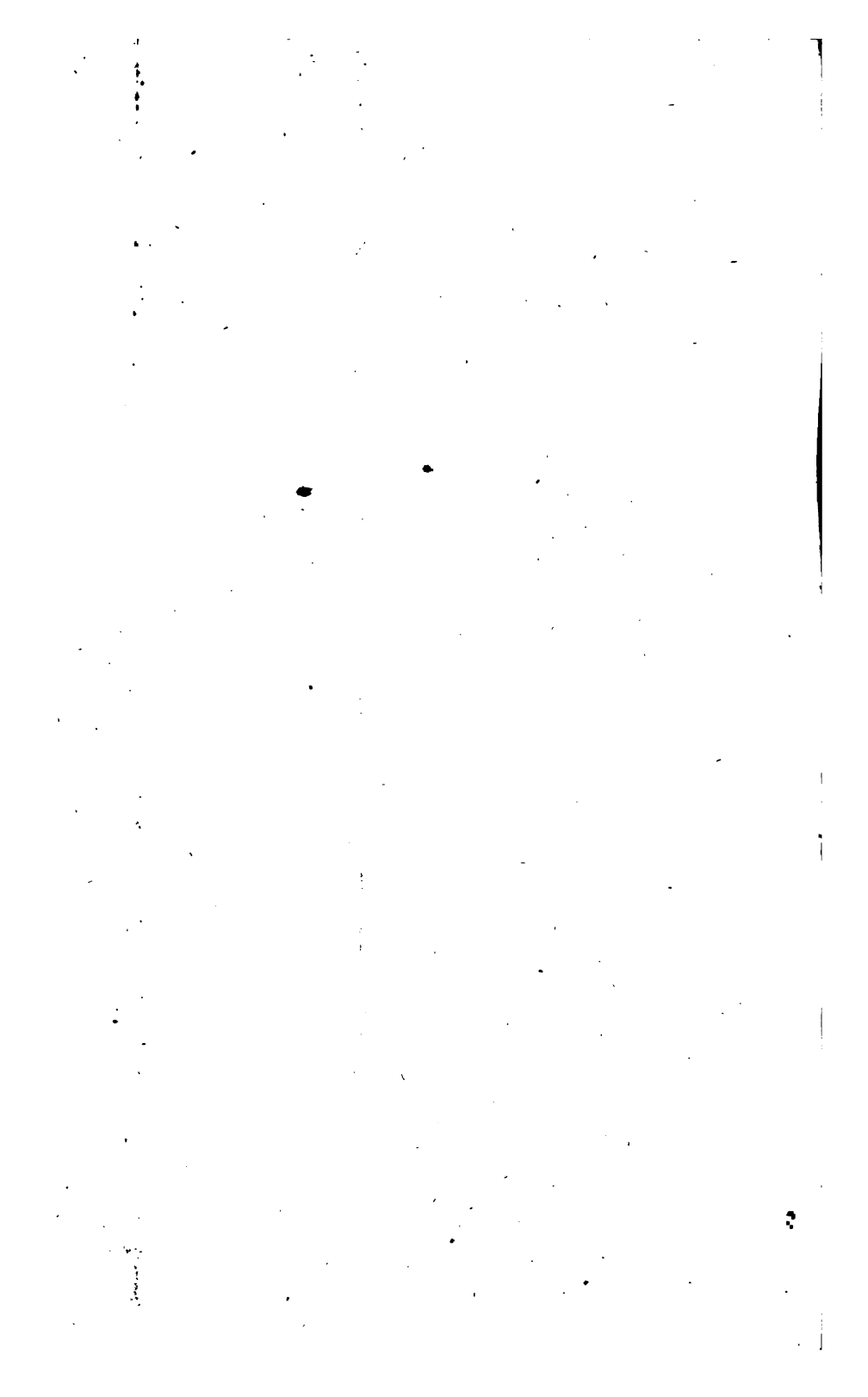
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NEW SERIES
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VOL. I. PART II.

ORIGINAL ESSAYS.

ARTICLE I.

DEMONSTRATION of some Propositions relating to such portions of the Surface and Solidity of a Sphere as may be exactly Squared and Cubed.

By Mr. JAMES IVORY, of the Royal Military College.

THE Problem proposed by Viviani, a Mathematician of Florence, to the Geometers of his time, is well known, and has been often resolved. Besides effecting what was required by Viviani, the construction given of the Florentine Problem leads to a number of curious properties; and, in addition to the properties that were known before, Bossut, in the second vol. of Memoirs of the National Institute, announces a New Theorem, resulting from the same construction, and determining a portion of the solidity of a sphere that may be exactly cubed, as the problem of Viviani determines a portion of the surface of a sphere that may be exactly squared.

The demonstrations that follow I believe to be new; and they are sufficiently simple.

1. PROB. "If a circle be described upon any radius of the base of an hemisphere, and upon the circle an upright cylinder be raised and be prolonged to pierce the hemisphere; it is required to find the surface of the oval hole, made in the spherical vault."

Assume any point in the curve formed by the intersection of the surfaces of the sphere and cylinder, and let x, y, z , be three co-ordinates determining the position of the assumed point; x , being perpendicular to the base of the hemisphere; y , perpendicular to the diameter of the base of the cylinder; and z , the part of that diameter intercepted between y and the centre of the hemisphere. Then if r denotes the radius of the hemisphere, we shall have $r^2 = x^2 + y^2 + z^2$. And because y is the ordinate and z the absciss of a semicircle of which the diameter is r , $y^2 = (r - z) \times z = rz - z^2$. From the latter equation we get $y^2 + z^2 = rz$, which being substituted in the former, there results $r^2 = x^2 + rz$.

Let a great circle pass through the assumed point and the pole of the hemisphere; and let ϕ denote the arch of it intercepted between the assumed point and the base of the hemisphere: then $x = r \sin. \phi$; $rz = r^2 - x^2 = r^2 \cos.^2 \phi$; therefore, $z = r \cos.^2 \phi$; and $y^2 = rz - z^2 = r^2 \cos.^2 \phi - r^2 \cos.^4 \phi = r^2 \cos.^2 \phi \sin.^2 \phi$; therefore, $y = r \cos. \phi \sin. \phi$.

Let a plane parallel to the base of the hemisphere be drawn through the assumed point; and let ρ denote the radius of the small circle which is the common section of the plane and sphere; then $\rho = r \cos. \phi$. Conceive also the plane of a great circle to pass by the axis of the cylinder and the pole of the hemisphere; it is obvious that this great circle will bisect the arch of the small circle lying within the oval curve on the surface of the sphere; it is plain too that y will be the sine of half the arch of the small circle to the radius ρ ; therefore, the sine of the

like arch, in the circle having unity for radius, is $\frac{y}{\rho} =$

$$\frac{r \sin. \phi \cos. \phi}{r \cos. \phi} = \sin. \phi. \text{ Consequently the measure of the half}$$

arch itself is ϕ ; and the measure of the whole arch is 2ϕ . Therefore, the length of the arch of the small circle, lying within the oval curve on the surface of the sphere is

$$2\phi \times \rho = 2r\phi \cos. \phi.$$

But the distance of the small circle from the base of the hemisphere, reckoned on a great circle passing through the pole, is $r \times \phi$; therefore, the fluxion of the part of the oval hole bounded by the arch of the small circle is $r\dot{\phi} \times 2r\phi \cos. \phi = 2r^2\dot{\phi}\phi \cos. \phi$: Consequently the part of the hole itself $= 2r^2 \times \int \phi \dot{\phi} \cos. \phi = 2r^2 \times (\phi \sin. \phi + \cos. \phi) + C$; to determine C we have $2r^2 + C = 0$, whence $C = -2r^2$: And the corrected fluent is $2r^2 \times (\phi \sin. \phi + \cos. \phi) - 2r^2$ which
expresses

expresses the surface of the part of the hole lying between the small circle and the base of the hemisphere.

Let $\pi = 3.1416$ &c. = semi-periphery to radius unity;
then putting $\phi = \frac{\pi}{2}$ in the above expression, we shall find the whole surface of the oval hole made in the spherical vault = $r^2\pi - 2r^2$.

2. THEOR. "If upon two radii that compose a diameter of the base of an hemisphere two circles be described; and upon the two circles two upright cylinders be raised and be prolonged to pierce the hemisphere; the portion of the surface of the hemisphere that remains after the two oval holes are taken away will be equal to four times the square of the radius of the hemisphere."

For, according to what we have just found, the surfaces of the two oval holes = $2r^2\pi - 4r^2$; but the whole surface of the hemisphere = $2r^2\pi$; therefore what remains, after the two holes are taken away, = $2r^2\pi - \{2r^2\pi - 4r^2\} = 4r^2$.

This theorem coincides with the result of the problem proposed by Viviani. Let the two cylinders be prolonged, below the plane of their bases, to pierce the other hemisphere; and let the plane of a great circle pass by the axes of the two cylinders. Then this great circle will divide the sphere into two hemispheres, and will bisect the four oval curves formed on the surface of the sphere; the four half-ovals that make part of the surface of each hemisphere are the four windows of Viviani; and the remaining part of the surface of the hemisphere is equal to $4r^2$ according to what has just been demonstrated.

3. THEOR. "The whole surface of the cylinder (1) included within the hemisphere is equal to $2r^2$."

Montucla in giving the History of the Florentine Problem, has noticed this property.

The fluxion of the base of the cylinder is $= \sqrt{\dot{z}^2 + \dot{y}^2}$
 $\left\{ \text{because } \dot{y} = r\dot{\phi} (\cos.^2 \phi - \sin.^2 \phi) \text{ and } \dot{z} = -2r\dot{\phi} \times \sin. \phi \right.$
 $\left. \cos. \phi \right\} = r\dot{\phi} \sqrt{(\cos.^2 \phi - \sin.^2 \phi)^2 + 4 \sin.^2 \phi \cos.^2 \phi} =$
 $r\dot{\phi} \sqrt{(\cos.^2 \phi + \sin.^2 \phi)^2} = r\dot{\phi}.$

But the fluxion of the surface of the cylinder bounded by
 $x = x\sqrt{\dot{z}^2 + \dot{y}^2} = r\dot{\phi} \times x = r^2\dot{\phi} \sin. \phi$; therefore, the surface
22
of

of the cylinder bounded by x , $= \int r^2 \dot{\phi} \sin. \phi = C - r^2 \cos. \phi$;
to determine C we have $0 = C - r^2$ and $C = r^2$; therefore,
the corrected fluent is $r^2 - r^2 \cos. \phi$; which expresses the
surface of the cylinder bounded by x and insisting on the arch
of the base lying between x and the point in which the base of
the cylinder touches the base of the hemisphere.

Put $\phi = \frac{\pi}{2}$; and the fluent becomes simply r^2 , which is
equal to the surface of the cylinder insisting on half the cir-
cumference of the base; therefore, the whole surface $= 2r^2$.

4. THEOR. "If two cones be formed, having their vertices
in the centre of the sphere, and their bases the two oval
curves (a) formed on the surface of the sphere; what
remains of the whole solidity of the hemisphere, after the
two cones are taken away, will be equal to $\frac{4}{3}r^3$."

For, since the spherical surface which is the base of either
cone has been shewn to be $= r^2\pi - 2r^2$, the solidity of the
cone $= \frac{1}{3}r \times (r^2\pi - 2r^2) = \frac{r^3\pi}{3} - \frac{2r^3}{3}$. Therefore, the
two cones $= \frac{2r^3\pi}{3} - \frac{4r^3}{3}$. But the whole solidity of the
hemisphere $= \frac{2r^3\pi}{3}$. Therefore, deducting the two cones, the
remainder $= \frac{4r^3}{3}$.

5. THEOR. "If from the whole portion of the cylinder
(1) included within the hemisphere, the cone, that makes
a part of it, be taken away, what remains will be equal
to $\frac{8}{3}r^3$."

For let a plane be drawn through the centre of the
sphere to pass by the ordinate x , and let s denote the
solidity of the part bounded by this plane, the surface of the
cylinder and the conical surface; from the point in which
 x meets the base of the hemisphere, let a tangent be drawn
to the base of the cylinder; and let p denote the perpendicular
drawn to the tangent from the centre of the sphere. Then s is
manifestly equal to a pyramid of which p is the altitude and the
fluxion

fluxion of the surface of the cylinder ($\pi x \sqrt{[z^2 + y^2]}$) the base: therefore, $\dot{s} = \frac{1}{3} p x \sqrt{[z^2 + y^2]}$.

But the double of the triangle having its vertex in the centre of the sphere, and $\sqrt{[z^2 + y^2]}$ for its base, is equal to $p \sqrt{[z^2 + y^2]}$. And the double of the same triangle is also equal to $(zy - yz)$. Consequently $p \sqrt{[z^2 + y^2]} = (zy - yz)$.

$$\begin{aligned} \text{Therefore, } \dot{s} &= \frac{1}{3} \times x (zy - yz) = \frac{r^3}{3} \sin. \phi \dot{\phi} \times (\cos.^3 \phi \\ &- \sin.^2 \phi \cos.^2 \phi + 2 \sin.^2 \phi \cos.^2 \phi) = \frac{r^3}{3} \sin. \phi \dot{\phi} \times \\ &(\cos.^4 \phi + \cos.^2 \phi \sin.^2 \phi) = \frac{r^3}{8} \times \sin \phi \cdot \dot{\phi} \times \cos.^2 \phi. \end{aligned}$$

Therefore, $s = C - \frac{r^3}{9} \cos.^3 \phi$: to determine C, we have

$$0 = C - \frac{r^3}{9} \text{ and } C = \frac{r^3}{9}: \text{ And } s = \frac{r^3}{9} - \frac{r^3}{9} \cos.^3 \phi;$$

where s denotes the solid bounded by the surfaces of the cone and cylinder, and insisting upon the part of the area of the base of the cylinder lying between the diameter and the chord drawn from the centre of the sphere through the point where x meets the base of the hemisphere.

When $\phi = \frac{\pi}{2}$, then $s = \frac{r^3}{9}$, which is the part of the solid insisting upon a semicircle of the base of the cylinder; therefore, the whole remaining part of the portion of the cylinder, where the cone is taken away, is equal to $\frac{2}{3} r^3$.

6. THEOR. "If from the whole solidity of the hemisphere, the portions of the two cylinders (a) included within it be taken away, what remains will be equal to $\frac{1}{3} \times (2r)^3$."

For we have already shewn (4) that the remainder of the hemisphere, after the two cones are taken away, is equal to $\frac{4}{3} r^3$; and if we further take away the sum of the excesses of the two cylindrical portions above the two cones (equal (5) to $\frac{4}{3} r^3$) the remainder will be $\frac{4}{3} r^3 - \frac{4}{3} r^3 = \frac{8}{3} r^3 = \frac{1}{3} \times (2r)^3$.

If the two cylinders be prolonged below the plane of their bases to pierce the other hemisphere, it is obvious that the portion taken away from the whole solidity of the sphere by the two cylinders, leaves a remainder equal to $\frac{2}{3} \times (2r)^3$; which is precisely the new theorem of Bossut.

7. Let w denote the length of the arch of the curve of double curvature formed on the surface of the sphere; then

$$\begin{aligned}
 \text{then } \dot{w} \sqrt{y^2 + z^2 + s^2} &= r \dot{\phi} \sqrt{(\text{cof.}^2 \phi - \text{fin.}^2 \phi)^2 + 4 \text{fin.}^2 \phi \text{cof.}^2 \phi + \text{cof.}^2 \phi} = r \dot{\phi} \sqrt{(\text{cof.}^2 \phi + \text{fin.}^2 \phi)^2 + \text{cof.}^2 \phi} \\
 &= r \dot{\phi} \sqrt{1 + \text{cof.}^2 \phi} = r \dot{\phi} \sqrt{2 - \text{fin.}^2 \phi} = \frac{r}{\sqrt{2}} \times \dot{\phi} \sqrt{1 - \frac{1}{2} \text{fin.}^2 \phi} : \text{Therefore, } w = \frac{r}{\sqrt{2}} \times \int \dot{\phi} \sqrt{1 - \frac{1}{2} \text{fin.}^2 \phi}.
 \end{aligned}$$

Now if an ellipse be taken, of which the two axes are 1 and $\frac{1}{\sqrt{2}}$: then $\int \dot{\phi} \sqrt{1 - \frac{1}{2} \text{fin.}^2 \phi}$ will denote the length of an arch of the ellipse, reckoned from the extremity of the conjugate axis, of which the ordinate, parallel to the transverse axis, is $\text{fin. } \phi$. Therefore, the rectification of the curve of double curvature depends upon the rectification of an ellipse having 1 and $\frac{1}{\sqrt{2}}$ for its axes.

ARTICLE II.

DEMONSTRATION of a Theorem respecting Prime Numbers.

By Mr. JAMES IVORY of the Royal Military College.

The following theorem, one of the most important in the theory of numbers, is the discovery of *Fermat*. It was published by him without demonstration, as indeed were many other of those curious properties of numbers that form the most difficult and interesting part of the *Diophantine Analysis*. The theorem was first demonstrated by Euler in the *Petersburgh Commentaries*; and his demonstration may be found in a work on the *Theory of Numbers*, published by Le Gendre*.

I believe the following demonstration to be new, and I reckon it more simple than that of Euler.

* *Essai sur la Théorie des Nombres*, Part 2de, Theor. 1re.

THEOREM.

Let p be any prime number, and N any number not divisible by p , then $N^{p-1} - 1$ will be divisible by p .

DEMONSTRATION.

Let there be taken all the multiples of N , viz.

$$N, 2N, 3N, 4N \dots\dots\dots (p-1)N$$

by the numbers in the natural series 1, 2, 3, 4, &c. continued as far as $(p-1)$, or so long as the numbers in the series are less than p ; and let us consider what will be the remainders if the several multiples be divided by p . That none of the multiples are divisible by p is very evident: for if $m \times N$ be one of them, p does not divide N by hypothesis, and m is less than p ; therefore, p cannot divide the multiple $m \times N$. I likewise say, that the several remainders are all different from one another. For if it be possible let $m \times N$ and $n \times N$ be two multiples that, when divided by p , have the same remainder r ; then

$$m \times N - r$$

$$n \times N - r$$

will be multiples of p : therefore, the difference $m \times N - n \times N = (m-n) \times N$ will likewise be a multiple of p : therefore, either $(m-n)$ or N must be a multiple of p , which is impossible; because $m-n$ is less than p , and N is not a multiple of p by hypothesis; therefore, no two of the multiples, when divided by p , can have the same remainder. But, from the nature of division, the remainders must be all integer numbers less than p , and as they are just as many as the integer numbers less than p , and have been shewn to be all different from one another; it follows, that the several remainders can be no other than the several integer numbers less than p , or than the several numbers in the series

$$1, 2, 3, 4 \dots\dots\dots (p-1).$$

Let us now adopt the notation $m \cdot p$ to denote a multiple of p in general, without attaching the notion of any particular number to the symbol m : then, because the dividend is a multiple of the divisor increased by the remainder, it is manifest from what has been proved that the multiples

$$N, 2N, 3N, 4N \dots\dots\dots (p-1)N$$

may be represented by

$$m \cdot p$$

$m \cdot p + 1, m \cdot p + 2, m \cdot p + 3, m \cdot p + 4, \dots (m \cdot p + (p-1))$; each number in the one series being equal to some one in the other series; therefore,

$$N \times 2N \times 3N \times 4N \dots \times (p-1)N = (m \cdot p + 1) \times (m \cdot p + 2) \times (m \cdot p + 3) \dots \times (m \cdot p + (p-1)).$$

Now, if the expression on the right-hand side of the equation be evolved, it will consist of a certain number of terms, all of which, excepting the last, will contain p , or some power of p , as a factor; and the last term will be the continued product of all the numbers less than p : let the last term be denoted by Q , then it is clear that the right-hand side of the equation may be represented by $m \cdot p + Q$: It is also obvious that the left-hand

side is equal to $Q \times N^{p-1}$: therefore $Q \times N^{p-1} = m \cdot p + Q$. Whence $Q \times N^{p-1} - Q = Q \times \{N^{p-1} - 1\}$

$= m \cdot p$. Therefore, p necessarily divides either Q or $N^{p-1} - 1$: But Q can have no divisors excepting numbers less than p or composite numbers formed by the product of numbers less than p multiplied together: consequently p cannot divide Q ; therefore it necessarily divides $N^{p-1} - 1$. Q. E. D.

COR. Hence if N be any number not divisible by the prime number p , the remainder of N^p when divided by p will be equal to the remainder of N divided by p .

For it has been shewn that $N^{p-1} - 1$ is always divisible by p : therefore, when N^p and N are divided by p , the remainders of the divisions must be the same.

ARTICLE III.

FAGNANI'S Theorem respecting Elliptic Arches, rendered more general. By Mr. JAMES IVORY, R. M. College.

THE curious theorem of Count *Fagnani*, which assigns the difference of two Arches reckoned from the extremities of the quadrant of an ellipse, is well known to Mathematicians: I propose at present to investigate a more general theorem, which comprehends that of *Fagnani* as a particular case, and which assigns the difference of two elliptic arches reckoned from any two given points in the periphery of an ellipse; and to deduce a similar property in all the conic sections.

1. Let ABC (fig. 96, pl. 5.) be a quadrant of an ellipse, of which the semi-axis major is 1, and the excentricity ϵ : let the elliptic arch BD be denoted by w and the ordinate DE being drawn,

put $CE = x$: then we shall readily obtain $w = \frac{R}{\epsilon} \sqrt{\frac{1 - \epsilon^2 x^2}{1 - x^2}}$;

or, putting $R = \sqrt{1 - \epsilon^2 x^2}$, $w = \frac{R}{\epsilon \sqrt{1 - x^2}}$.

In like manner if the arch $Bd = w'$; $Ce = x'$; and $R' = \sqrt{1 - \epsilon^2 x'^2}$: then $w' = \frac{R'}{\epsilon \sqrt{1 - x'^2}}$.

Now $\frac{R^2 - R'^2}{x'^2 - x^2} = \epsilon^2$: and if we put $y = x' \sqrt{1 - x^2} + x \sqrt{1 - x'^2}$, and $z = x' \sqrt{1 - x^2} - x \sqrt{1 - x'^2}$; then $y \times z = x'^2 - x^2$: consequently $\frac{R^2 - R'^2}{y \times z} = \frac{R + R'}{y} \times \frac{R - R'}{z} = \epsilon^2$.

Suppose that the variable elliptic arch BD or w becomes BM ($=m$) when the variable elliptic arch Bd or w' becomes BN ($=n$); and put p and q to denote CP & CQ, which are the corresponding values of CE & Ce or of x & x' : then as the formula last investigated is true, whatever x & x' denote it will be true, when p and q are substituted for x & x' ; therefore, attending to what R , R' , y & z were put to denote, and if we further put

$$\left. \begin{aligned} a &= \frac{\sqrt{1 - \epsilon^2 p^2} + \sqrt{1 - \epsilon^2 q^2}}{q \sqrt{1 - p^2} + p \sqrt{1 - q^2}} \\ b &= \frac{\sqrt{1 - \epsilon^2 p^2} - \sqrt{1 - \epsilon^2 q^2}}{q \sqrt{1 - p^2} - p \sqrt{1 - q^2}} \end{aligned} \right\} \dots\dots\dots \text{No. 1.}$$

it will easily appear that we shall have $a \times b = \varepsilon^2$.

Therefore, $\frac{R + R'}{y} \times \frac{R - R'}{z} = a \times b$: and as each side

of this equation is identically equal to ε^2 , it establishes no relation between the variable quantities x & x' , but leaves them quite independent. We are therefore at liberty to determine one of the two variable quantities x & x' , by means of the other according to any relation that may best suit our purpose. Let us then suppose $R + R' = ay$; then on account of the preceding formula we shall necessarily have $R - R' = bz$. Hence by substituting for y and z , their values in x & x' , we get

$$\begin{aligned} 2R &= (a + b) x' \sqrt{(1 - x^2)} + (a - b) x \sqrt{(1 - x'^2)} \quad \text{No. 2.} \\ 2R &= (a - b) x' \sqrt{(1 - x^2)} + (a + b) x \sqrt{(1 - x'^2)} \end{aligned}$$

The equations just obtained, being the one a necessary consequence of the other, only involve a certain relation between the two variable quantities x & x' ; and I shall now endeavour to express this relation in a formula as simple as possible. For this purpose, let the equations, No. 2, be squared and added together: then $2R^2 + 2R'^2 = (a^2 + b^2) \cdot \left\{ x^2 (1 - x'^2) \right.$

$$\left. + x'^2 (1 - x^2) \right\} + 2(a^2 - b^2) \times x x' \sqrt{\{(1 - x^2)(1 - x'^2)\}} \\ = 4 - 2\varepsilon^2(x^2 + x'^2) = 4 - 2ab(x^2 + x'^2).$$

$$\begin{aligned} \text{Hence, } 4 &= (a + b)^2 \times \left\{ x'^2 (1 - x^2) + x^2 (1 - x'^2) \right\} \\ &- (a - b)^2 x^2 x'^2 + 2(a^2 - b^2) \times x x' \sqrt{\{(1 - x^2)(1 - x'^2)\}} \end{aligned}$$

Subtract both sides from $(a + b)^2$ and take the square root; then

$$(a + b) \sqrt{\{(1 - x^2)(1 - x'^2)\}} - (a - b) x x' = \sqrt{\{(a + b)^2 - 4\}} \quad \text{No. 3.}$$

which appears to be the simplest formula expressing the required relation of the variable quantities x & x' .

Multiply the first of the equations, No. 2, by $\frac{x}{\sqrt{(1 - x^2)}}$ and the second by $\frac{x'}{\sqrt{(1 - x'^2)}}$: then, having substituted $\dot{x}$ for

$\frac{R\dot{x}}{\sqrt{(1 - x^2)}}$ and $\dot{x}'$ for $\frac{R'\dot{x}'}{\sqrt{(1 - x'^2)}}$ and added the two equations together, there will result

$$m + m' = \frac{a + b}{2} (x'\dot{x} + x\dot{x}') + \frac{a - b}{2} \times \left\{ \frac{x\dot{x}\sqrt{(1 - x'^2)}}{\sqrt{(1 - x^2)}} + \frac{x'\dot{x}'\sqrt{(1 - x^2)}}{\sqrt{(1 - x'^2)}} \right\}$$

and

and taking the fluents

$$w + w' = \frac{a+b}{2} \times x'x - \frac{a-b}{2} \cdot \sqrt{\{(1-x^2)(1-x'^2)\}} + C,$$

an expression that may be rendered more simple by exterminating the radical on the right-hand side by means of the formula, No. 3; it will then become

$$w + w' = \frac{2ab \cdot x x'}{a+b} - \frac{1}{2} \cdot \frac{a-b}{a+b} \sqrt{\{(a+b)^2 - 4\}} + C.$$

To determine C, we have only to remark that when $w = m$, $w' = n$ and $p = x$ and $q = x'$: therefore,

$$m + n = \frac{2ab}{a+b} \cdot pq - \frac{1}{2} \cdot \frac{a-b}{a+b} \sqrt{\{(a+b)^2 - 4\}} + C:$$

consequently, writing ϵ^2 for $a \times b$, we have

$$(w - m) - (n - w') = \frac{2}{a+b} \times \epsilon^2 (xx' - pq) \dots \dots \dots \text{No. 4.}$$

that is, the difference of the elliptic arches MD and Nd, reckoned from the two given points M and N, is equal to the algebraic quantity

$$\frac{2}{a+b} \times \epsilon^2 (xx' - pq): \text{ where } a \text{ and } b \text{ are to be}$$

determined from p and q by the formula, No. 1; and the relation of x and x' is expressed by the formula, No. 3.

If we suppose M to coincide with B, and N with A, (fig. 97, pl. 5,) then $p = 0$, $q = CA = 1$; and the formulas, No. 1, give $a = 1 + \sqrt{(1 - \epsilon^2)}$, $b = 1 - \sqrt{(1 - \epsilon^2)}$; and the formula, No. 3,

$$\text{gives } \sqrt{(1 - \epsilon^2)} = \sqrt{\left\{ \frac{(1 - x^2)(1 - x'^2)}{xx'} \right\}}: \text{ and in this case the}$$

difference of the two elliptic arches BD & Ad, reckoned from the extremities of the quadrant, is equal to $\epsilon^2 \times xx'$; and this will be found to be nothing more than Fagnani's well known Theorem.

If x be taken equal to x' ; so that the points D & d come together in H: then $\sqrt{(1 - \epsilon^2)} = \frac{1 - x^2}{x^2}$, therefore $x^2 = CK^2 =$

$$\frac{1}{1 + \sqrt{(1 - \epsilon^2)}}; \text{ and the difference of the arches BH \& HA} =$$

$$\epsilon^2 x^2 = \frac{\epsilon^2}{1 + \sqrt{(1 - \epsilon^2)}} = 1 - \sqrt{(1 - \epsilon^2)} = \text{the difference of the semi-axis of the ellipse.}$$

2d. Let us now proceed to the hyperbola. Let C (fig. 98, pl. 5.) be the centre, and CA half the greater axis of an hyperbola:

Let w & w' denote the variable arches AD , $A'd'$, and x & x' the distances CE & $C'e$ of the corresponding ordinates from the centre: then if the semi-axis AC be 1, & the excentricity or distance of the

focus from the centre be ε ; we shall have $\dot{w} = \frac{\dot{x} \sqrt{(\varepsilon^2 x^2 - 1)}}{\sqrt{(x^2 - 1)}}$

and $\dot{w}' = \frac{\dot{x}' \sqrt{(\varepsilon^2 x'^2 - 1)}}{\sqrt{(x'^2 - 1)}}$: or writing R & R' for $\sqrt{(\varepsilon^2 x^2 - 1)}$ and

$$\sqrt{(\varepsilon^2 x'^2 - 1)}; \dot{w} = \frac{R \dot{x}}{\sqrt{(x^2 - 1)}} \text{ and } \dot{w}' = \frac{R' \dot{x}'}{\sqrt{(x'^2 - 1)}}$$

It is manifest that $\frac{R^2 - R'^2}{x^2 - x'^2} = \varepsilon^2$: and if we put $y = x' \sqrt{(x^2 - 1)}$

+ $x \sqrt{(x'^2 - 1)}$ and $z = x' \sqrt{(x^2 - 1)} - x \sqrt{(x'^2 - 1)}$ we shall

likewise have $y \times z = x^2 - x'^2$: therefore $\frac{R + R'}{y} \times \frac{R - R'}{z}$

$$= \varepsilon^2.$$

Suppose that when the point D coincides with M , that d coincides with N : put $AM = m$ and $AN = n$ and let p and q denote CP and CQ which are the contemporary values of x and x' : Let

$$\left. \begin{aligned} a &= \frac{\sqrt{(\varepsilon^2 p^2 - 1)} + \sqrt{(\varepsilon^2 q^2 - 1)}}{q \sqrt{p^2 - 1} + p \sqrt{q^2 - 1}} \\ b &= \frac{\sqrt{(\varepsilon^2 p^2 - 1)} - \sqrt{(\varepsilon^2 q^2 - 1)}}{q \sqrt{p^2 - 1} - p \sqrt{q^2 - 1}} \end{aligned} \right\} \dots\dots\dots \text{No. 1.}$$

then $a \times b = \varepsilon^2$.

Therefore, $\frac{R + R'}{y} \times \frac{R - R'}{z} = a \times b$; a formula which

holds good whatever x , x' , p and q , denote. Now, by pursuing the reasoning exactly in the same manner as has been done in the case of the ellipse, we shall find

$$\left. \begin{aligned} 2 R &= (a + b) x' \sqrt{(x^2 - 1)} + (a - b) x \sqrt{(x'^2 - 1)} \\ 2 R' &= (a - b) x' \sqrt{(x^2 - 1)} + (a + b) x \sqrt{(x'^2 - 1)} \end{aligned} \right\} \dots\dots\dots \text{No. 2.}$$

where x and x' have to one another the relation expressed in the formula.

$$a + b \times \sqrt{\{(x^2 - 1)(x'^2 - 1)\}} + (a - b)x = \sqrt{\{(a + b)^2 - 4\}} \dots\dots\dots \text{No. 3.}$$

Let the first of the equations, No. 2, be multiplied by $\frac{\dot{x}}{\sqrt{(x^2 - 1)}}$

and the second by $\frac{\dot{x}'}{\sqrt{(x'^2 - 1)}}$; then having added the results together,

gether, we shall find an equation that may be integrated, and of which the fluent is

$$w + w' = \frac{a+b}{2} x x' + \frac{a-b}{2} \sqrt{(x^2-1)(x'^2-1)} + C \dots \text{No. 4.}$$

from this equation the quantity C will be exterminated by means of m , n , p and q , which are contemporary values of w , w' , x and x' : and if, for the sake of simplicity, we further exterminate the radical quantity by means of the formula, No. 3, we shall finally obtain

$$(w' - m) - (n - w') = \frac{2}{a+b} \times \epsilon^2 (x x' - p q) \dots \text{No. 5.}$$

that is, the difference of the hyperbolic arches MD and Nd, reckoned from the given points M and N, and corresponding to x and x' of which the valuation is expressed by the formula, No. 3,

is equal the algebraic quantity $\frac{2}{a+b} \times \epsilon^2 \times (x x' - p q)$.

The property that we have just investigated for the hyperbola is equally simple with that before found for the ellipse, and is quite analogous to it. We have thus another property, in addition to so many others already known, common to the two curves, and admitting of an enunciation in the same words. Many curious properties may be deduced from the expressions, marked No. 4 and No. 5: but I shall here confine my attention to one only, which effects for the hyperbola what *Fagnani's* theorem does for the ellipse. The property I allude to is, in effect, contained in the formula, No. 4, and may be derived from it by making p and q equal to their extreme limits; that is, by taking p infinitely great and $q = 1$: but, as this method of deduction would introduce notions that, in strictness, may be objected to, I shall give a rigid investigation of it.

Resume the formula $\frac{R + R'}{y} \times \frac{R - R'}{z} = \epsilon^2$ which supposes

no relation between x and x' ; and let us suppose that these variable quantities are so related that $R + R' = \epsilon y$; then, of consequence, $R - R' = \epsilon z$; whence, by restoring the values of y and z :

$$\begin{aligned} R &= \epsilon x \sqrt{x^2 - 1} \\ R' &= \epsilon x' \sqrt{x'^2 - 1}. \end{aligned}$$

And we shall readily find that the relation of x and x' , which these formula involve, may be thus simply expressed;

$$\epsilon \sqrt{(x^2 - 1)(x'^2 - 1)} = \sqrt{(\epsilon^2 - 1)} \dots \dots \dots \text{No. 6.}$$

Multiply

Multiply the first of the two formulas by $\frac{x}{\sqrt{(x^2 - 1)}}$, and the second by $\frac{x'}{\sqrt{(x'^2 - 1)}}$; add the results together; take the fluents; and there will come out

$$w + w' + C = \varepsilon x x' \dots \dots \text{No. 7.}$$

Draw the asymptote of the hyperbola, (fig. 99, pl. 5.) and produce the ordinates DE, *de* to meet it in S and R: draw AF parallel to the ordinates; take CT a fourth proportional to CF ($=\varepsilon$), CR and CS, and the formula, No. 7, amounts to this, "That the excess of the line CT, above the sum of the two hyperbolic arches Ad, AD, is a constant quantity C, provided x and x' satisfy the formula, No. 6."

The formula, No. 6, shews that x becomes indefinitely great when x' approaches indefinitely to unity: whence it is easy to infer that the constant quantity C is the excess of the whole asymptote infinitely produced above the whole hyperbolic curve infinitely continued.

The constant quantity C will be most simply determined from the case when $x = x'$: in this case the formula, No. 6, gives

$$x^2 = \frac{\varepsilon + \sqrt{(\varepsilon^2 - 1)}}{\varepsilon} \text{ whence this construction. "Take FG}$$

$=$ FA, and CL a mean proportional between CF and CG: then, the constant quantity C, or the excess of the hyperbolic curve above the asymptote, both infinitely continued, will be equal to the excess of the line CG, above twice the hyperbolic arch AH."

I add one more theorem. "Having determined the points H and G as above, let $de \times DE = HK^2$; and take CT a fourth proportional CF, CR and CS: then the excess of the arch HD above the arch Hd will be equal to the straight line GT: or, which is the same thing, the excess of the sum of the arches AD and Ad, above twice the arch AH, will be equal to the straight line GT."

By this Theorem the rectification of an Arch AD greater than AH, is reduced to the rectification of another arch Ad less than AH.

3d. We proceed now to the parabola. Let AE (fig. 100, pl. 5.) be the axis: put $AE = x$, $Ae = x'$ and let w and w' denote the corresponding arches of the parabola: then, 4ε denoting the parameter of the axis, we have, $w = \sqrt{\frac{x}{\varepsilon}} \sqrt{(\varepsilon + x)}$ and w'

$$= \sqrt{\frac{x'}{\varepsilon}} \sqrt{(\varepsilon + x')}; \text{ or, writing R and R' for the radicals}$$

$\sqrt{(\varepsilon + x)}$

$\sqrt{(\varepsilon + x)}$ and $\sqrt{(\varepsilon + x')}$, $\dot{w} = \frac{R\dot{x}}{\sqrt{x}}$, and $\dot{w}' = \frac{R'\dot{x}'}{\sqrt{x'}}$.

It is manifest that $\frac{R^2 - R'^2}{x - x'} = \frac{R + R'}{\sqrt{x} + \sqrt{x'}} \times \frac{R - R'}{\sqrt{x} - \sqrt{x'}} = 1$, whatever x and x' denote. Suppose $AM (=m)$ and $AN (=n)$ to be contemporary values of w , & w' and $AP = p$ and $AQ = q$ to be the corresponding values of x and x' : put

$$\left. \begin{aligned} a &= \frac{\sqrt{(\varepsilon + p)} + \sqrt{(\varepsilon + q)}}{\sqrt{p} + \sqrt{q}} \\ b &= \frac{\sqrt{(\varepsilon + p)} - \sqrt{(\varepsilon + q)}}{\sqrt{p} - \sqrt{q}} \end{aligned} \right\} \dots\dots\dots \text{No. 1.}$$

then $a \times b = 1$: and $\frac{R + R'}{\sqrt{x} + \sqrt{x'}} + \frac{R - R'}{\sqrt{x} - \sqrt{x'}} = a \times b$.

The formula just found supposes no relation between the variable quantities x and x' : we may, therefore, suppose $R + R' = a\sqrt{x} + a\sqrt{x'}$; and it will follow that $R - R' = b\sqrt{x} - b\sqrt{x'}$: whence

$$\begin{aligned} 2R &= (a + b)\sqrt{x} + (a - b)\sqrt{x'} \\ 2R' &= (a - b)\sqrt{x} + (a + b)\sqrt{x'} \end{aligned}$$

two formulas that involve a relation of x and x' , more simply expressed by the equation

$$4\varepsilon = (a - b) \times \left\{ a(\sqrt{x} + \sqrt{x'})^2 - b(\sqrt{x} - \sqrt{x'})^2 \right\} \text{No. 2}$$

Multiply the two formulas for R and R' by $\frac{\dot{x}}{\sqrt{x}}$ and $\frac{\dot{x}'}{\sqrt{x'}}$ respectively, and having added them together we shall still find an equation that is integrable, and of which the fluent is

$$w + w' = C + \frac{a + b}{2} \times (x + x') + (a - b) \times \sqrt{(xx')} \dots \text{No. 3.}$$

and, if we further exterminate the radical $\sqrt{(xx')}$ by means of the formula, No. 2, and change the constant quantity we shall find

$$w + w' = \frac{2}{a + b} \times (x + x') + C \dots\dots\dots \text{No. 3.}$$

In this formula C will be determined by any contemporary values of w , w' , x and x' , as m , n , p and q : thus we have

$$(w - m) - (n - w') = \frac{2}{a + b} \times \{ (\cancel{a}p) - (q - \cancel{x}') \} \text{No. 4.}$$

That is the difference of the parabolic arches MD and ND is to the difference of PE and cQ as $(a + b)$ to 2.

I shall not prosecute this subject any further at present, but may resume it on another occasion.

ARTICLE

ARTICLE IV.

A GEOMETRICAL PORISM, *with two Examples of its application to the Solution of Problems.* By SCOTICUS.

THE following Porism is extremely simple and obvious; it affords, however, a good example of the peculiar nature of those propositions, and of their use in the solution of geometrical problems.

PORISM. Fig. 101. Pl. 5.

Let A and B be two given points, and CDE a circle given by position, there is a given point P in the straight line joining the points A, B, such, that if from P any straight line whatever be drawn, meeting the circle in C and D, the points A, B, C, D, shall be in the circumference of a circle.

Suppose the porism to be true; take C' any given point in the circumference of the given circle, and join C'P meeting the circle again in D'. Because the lines PDC, PD'C' meet the circle, $CP \cdot PD = C'P \cdot PD'$; but by hypothesis the points A, B, C, D, are in the circumference of a circle, and, therefore, $CP \cdot PD = AP \cdot PB$, therefore, $AP \cdot PB = C'P \cdot PD'$, so that the points A, B, C', D', are also in the circumference of a circle; now A, B, C' are given points, therefore, the circle passing through them and D' is given by position, and hence the point D' and the line CD' are given by position; therefore P, the point in which it meets AB, is given, as was to be shewn.

CONSTRUCTION. Take any point C' in the circumference of the given circle, and describe a circle through the points A, B, C', meeting the given circle again in D'; join CD', meeting AB in P, the point required; that is, if from P, any straight line be drawn, meeting the circle in C and D, the points A, B, C, D, will be in the circumference of a circle. The synthetical demonstration is sufficiently evident, as also the limitation of the data.

PROBLEM I.

(Simpson's Geometry, Problem 45.) Fig. 101. Plate 5.

To describe a circle that shall pass through two given points A, B, and touch another circle C, E, D, given by position.

CONSTRUCTION.

CONSTRUCTION. Take any point C' in the circumference of the given circle, and describe a circle through the given points A, B, C' , which will touch the given circle in a point D , (in which case the thing required is done,) or will cut it again in a point D' ; join $C' D'$ meeting AB in P , and draw PE touching the given circle in E ; a circle described through A, B, E , is evidently that which is required; the demonstration is evident.

PROBLEM II.

(Simpson's Geometry, Prob. 15, or Select Exercises, Prob. 49.)
Fig. 102. Plate 5.

From two given points A, B , to draw two straight lines AD, BD to meet in a straight line MN , given by position, so that $AD + BD$, may be equal to a given line.

CONSTRUCTION. On B , one of the given points as a centre, with a radius equal to $AD + BD$, the given line, describe a circle; from A , the other point, draw AH perpendicular to MN , and produce the perpendicular so that $HK = HA$; through the points A, K , describe any circle whatever that may meet the other circle in F and G ; join GF meeting AK in C , and draw CE , also Ce touching the circle whose centre is B in E and e ; join BE, Be , meeting MN in D and d , and draw AD, Ad ; then AD, BD , also Ad, Bd , are the lines required.

DEMONSTRATION. Because $KC \cdot CA = GC \cdot CF = EC^2$, the line EC is a tangent to a circle passing through A, K, E , therefore, its centre is in the line EB ; but because MN bisects AB at right angles, its centre is also in MN , therefore, it is at D the intersection of EB and MN ; hence $AD = DE$, and $AD + DB = EB$, the given line. In the same way it may be shewn that $Ad + dB =$ the given line.



ARTICLE V.

GEOMETRICAL PROPOSITIONS.

By Mr. IVORY, R. M. College.

THE Propositions of which I propose to give the Geometrical Analysis, were published without Solutions by Dr. STEWART, at the end of his GENERAL THEOREMS: and they were given by him as some of the properties of the circle that had occurred in the investigation of those remarkable propositions. They are

are of that class called by the ancients, Porisms; the nature of which, before very obscure, has been fully elucidated by the learned Professor Playfair, in a memoir that completes the restoration of the most difficult branch of the ancient Geometry.

PROPOSITION I. PORISM. Fig. 103. Pl. 5.

Let there be a circle given by position, and let A, B, be two given points: a point C may be found within the circle, such, that if through the point C there be drawn any line meeting the circle in D and E, and AD, BD, AE, BE, be joined; the rectangle ADB will be to the rectangle AEB as CD to CE.

ANALYSIS.

Take O, the centre of the circle, and join OB: draw GH through C at right angles to BO, and join AG, GB, AH, HB: join also AC, and let it meet BO in F. Then, because GH is drawn through C, by hypothesis,

$$AG \times GB : AH \times HB :: GC : CH;$$

but, because GH is at right angles to BO passing through the centre of the circle, it will be bisected by BO: therefore, $BG = BH$, as it is easy to prove: therefore,

$$AG : AH :: GC : CH.$$

Therefore, the line AC will bisect the angle GAH. Suppose a circle to be described about the triangle GAH: then, because the angles GAF, HAF are equal, AF will pass through the middle point of the arch of the circle subtended by GH: and because BO bisects GH at right angles, B will pass through the same point: therefore, F, cannot but be the middle point of that arch. Therefore, $AC \times CF = GC \times CH = DC \times CE$.

From what has been shewn, it follows that the points A, E, F, D, are in the circumference of a circle: let the circle be described, and let FK be a diameter: draw DP, EQ, perpendicular to AF. Then, (Prop. C. VI. Simson's Euclid) $AD \times DF = FK \times DP$ and $AE \times EF = FK \times EQ$:

$$\text{therefore, } AD \times DF : AE \times EF :: DP : EQ :: DC : CE.$$

But, by hypothesis,

$$AD \times DB : AE \times EB :: DC : CE;$$

therefore,

$$AD \times DF : AE \times EF :: AD \times DB : AE \times EB.$$

Hence,

Hence, it is easy to infer, that
 $FD : is DB :: FE : EB.$

Attending now to what is demonstrated in the analysis of the second Proposition of the second Book of the *Loci Plani* of Apollonius, we must conclude from the last analogy, that the rectangle $BO \times OF$, is equal to the square of the semidiameter of the given circle. Therefore, the point F is given; and the line AF is given by position. Since then A and F are given points, and since what has been shewn in regard to DE is true of any chord passing through C , we may assume the point D any where in the circumference of the given circle: then the circle ADE will be determined, and of consequence the point E : therefore the point C , which is the intersection of AF and DE , will be given.

Hence this construction: draw BO to the centre of the given circle; take OF a third proportional to BO , and the semidiameter, and join AF ; assume any point in the circumference of the given circle (not in the same straight line that the points A and F are), and through A, D, F , describe a circle cutting the given circle again in E : join DE , cutting AF in C , and C will be the point sought.

The curious property of the point investigated in this Proposition may be made more general, as in the following Theorem. In order to render the demonstration more simple a lemma is premised.

PROPOSITION II. LEMMA. Fig. 104. Pl. 5.

In the triangle ACB , let CD bisect the angle ACB , and let AE and AF be drawn to make equal angles (BAE, CAF) with the sides AB and AC , and to meet CD in E and F : then BE and BF being drawn, AE will be to EB as AF to FB .

From F and E draw the perpendiculars FG, FH, FN, EK, EL , and EM : then because CD bisects the angle ACB , it is obvious that $FG = FH$ and $EK = EL$. Because the angle BAE is equal to CAF ; therefore the triangle MAE will be similar to the triangle FAG ; and the triangle NAF to EAK : therefore,

$$AE : EM :: AF : FG$$

$$AE : EK :: AF : FN.$$

Consequently,

$$EM : EK :: FG : FN;$$

that is,

$$EM : EL :: FH : FN.$$

Now the angles MEL , and NFH , are likewise equal to one another,

another, for each of them is the supplement of the angle at B: therefore the triangle MEL is similar to HFN, and, consequently, the angle HNF is equal to the angle MLE. But the angle HNF is equal to HBF, and the angle MLE to MBE, as is obvious: therefore the angle HBF is equal to MBE, and the triangles FBH and EBM are similar to one another;

Therefore,

$$EB : EM :: FB : FH ;$$

but as was before shewn,

$$AE : EM :: AF : FG \text{ or } FH.$$

Therefore,

$$AE : EB :: AF : FB \dots\dots\dots Q. E. D.$$

Cor. If AE and AF be drawn, to make equal angles with BA and AC. Then BE and BF will make equal angles with BA and BC.

PROPOSITION III. THEOREM. Fig. 105. Pl. 5.

Let there be a circle, of which O is the centre, and two points, A and B, as in Prop. 1.; and let C be the point found in that Proposition: draw AO, OB, to the centre, and through C draw MN parallel to OT, the line that bisects the angle AOB: let CD, CE, be any how drawn from C to meet the circle in D and E, and to make equal angles with MN, and join DA, DB, EA, EB, then the rectangle ADB is to the rectangle AEB as CD to CE.

DEMONSTRATION.

Let RS be drawn to bisect AF at right angles, and let it meet OB in S: draw OH at right angles to DC, and OG at right angles to EC, and let OH and OG meet RS in H and G; join AH, AG, AS, DF and EF: draw OR perpendicular to RS, and DP, EQ to AF: and let DC, and EC, meet the circle again in *d, e*.

Then, because DC, EC, make equal angles with MN, therefore OH and OG, which are perpendicular to DC, CE, will likewise make equal angles with MN, or with OT parallel to MN: and because the angles AOT, BOT, are equal, as well as the angles HOT, GOT, therefore the angle AOH is equal to the angle BOG. Again, because RS bisects AF at right angles, it is easy to infer that AS will be equal to SF, and that the angle ASO will be bisected: therefore applying to the triangle AOS what was demonstrated in the preceding lemma, it will be

$$OH : HA :: OG : GA ;$$

whence it is easy to shew, that

$$OH \times DP : OG \times EQ :: 2 HA \times DP : 2 GA \times EQ.$$

From

From the nature of the point C, the four points A, D, F, *d*, are in the circumference of a circle: now the centre of that circle must be in the line OH, which is perpendicular to D*d*, and bisects D*d*; and the same centre must likewise be in the line RS, because RS bisects AF at right angles: therefore, H the concurrence of OH and RS, must be the centre of the circle passing through the point A, D, F, *d*. Therefore, 2HA is equal to a diameter of the circle passing through A, D, F, *d*; and the rectangle 2HA × DP is equal to AD × DF (Prop. C. VI. Simfon's Euclid). In like manner it is shewn, that 2GA × EQ = AE × EF. Therefore, on account of the analogy before demonstrated, OH × DP : OG × EQ :: AD × DF : AE × EF.

Because OH is perpendicular to DC, and RS to AF, therefore the angle DCA is equal to the angle OHR, and the triangle DCP is similar to OHR. Therefore OH : OR :: CD : DP, and, consequently, OH × DP = OR × CD. In like manner it is shewn that OG × EQ = OR × EC.

Therefore,

OH × DP : OG × EQ :: OR × CD : OR × CE :: CD : CE.
And combining this, with what is shewn above,

$$AD \times DF : AE \times EF :: CD : CE.$$

Because, BO × OF, is equal to the square of the semidiameter of the circle, it follows, somewhat is demonstrated in the second Proposition of the second Book of the *Loci Plani* of Apollonius, that

$$FD : DB :: FE : EB.$$

Hence,

$$AD \times DF : AE \times EF :: AD \times DB : AE \times EB.$$

But, as has been shewn,

$$AD \times DF : AE \times EF :: CD : CE;$$

therefore,

$$AD \times DB : AE \times EB :: CD : CE.$$

Q. E. D

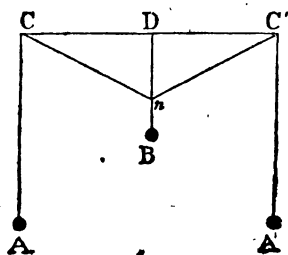
(To be continued.)

ARTICLE VI.

Solution of a Dynamical Question, by Mr. IVORY.

THE following question in Dynamics is Prop. 3d. page 131, of Simpson's Miscellaneous Tracts; and it is also resolved in Mr. Atwood's Treatise on Rectilineal Motion: but the solutions of these two authors bring out very different results. Mr. George Sanderfon, of London, having pointed out the above circumstance to Mr. Leybourn (in a letter, an extract from which was shewn to me), I was led to consider the problem; and, in consequence, found out the solution I am now to explain. The question is thus proposed by Simpson:

"Suppose that a thread, $ACnC'A'$, having two equal weights, A, A' , suspended at the ends thereof, is hung over two tacks, C, C' , in the same horizontal line; and that to the middle point of the thread (n), equally distant from the tacks, another given weight B is fixed, which is permitted to descend by its own gravity, so as to cause the other two weights, at the same time, to ascend; it is proposed to find the law of the velocity by which the said weights ascend and descend, abstracting from the resistance of the air the weight of the thread, and the friction on the tacks."



SOLUTION.

Draw nD perpendicular to the horizontal line CC' ; then it is obvious that $CD = C'D$. Let $CD = a$, $Dn = x$, and $Cn = y$: also let $g = 32\frac{1}{2}$, the measure of the accelerating force of gravity. Put τ to denote an indefinitely small interval, or the fluxion of the

the time, which is supposed to flow uniformly : then the velocity of the weight A in ascent $= \frac{\dot{y}}{\tau}$: and the variation of the velocity

$= \frac{\ddot{y}}{\tau^2}$. But the weight A cannot ascend at all, unless its gravity

be counteracted, and an additional force be impressed upon it equivalent to produce the actual velocity in ascent. Therefore, the force (constantly acting on each particle of A), necessary to

cause A to ascend with the velocity $\frac{\dot{y}}{\tau}$ is measured by $\left(g + \frac{\ddot{y}}{\tau}\right)$

and the whole motive force acting on A is, therefore, $= \left(g + \frac{\ddot{y}}{\tau}\right) \times$

A, an expression that, for the sake of brevity, we shall denote by F. Now the motive force F, by which A is urged upward, being derived from the action of the weight B ; therefore B will be reacted upon in the direction nC by an equal motive force, F. But the force in the direction nC may be resolved into two others : one

$= \frac{CD}{Cn} \times F = \frac{a}{y} \times F$, urging B in a direction parallel to DC ; and

another force $= \frac{nD}{nC} \times F = \frac{x}{y} \times F$, urging B in the direction

nD, and retarding the descent of B.

Because the weight A is equal to the weight A', and that each of them acts in a similar manner on the weight B : therefore, the effect of the weight A', is, in like manner, resolved into two forces acting on B ; one in the direction DC', and precisely equal to the former force in the direction DC, and, therefore, destroying its effect : and another force in the direction nD, likewise equal to the former force in that direction, but conspiring with it to retard the descent of B. Therefore, the whole effect of the two weights A and A' on the weight B, is equivalent to a motive

force $= 2 \times \frac{x}{y} \times F = \frac{x}{y} \times \left(g + \frac{\ddot{y}}{\tau}\right) \times 2A$, directly opposing the descent of B.

Again, the velocity with which B actually descends is $= \frac{\dot{x}}{\tau}$ and the variation of that velocity, or the accelerating force with which B actually descends $= \frac{\ddot{x}}{\tau}$. Therefore the accelerating

force

force constantly destroyed by the connection of B with the weight A, $A' = g - \frac{\ddot{x}}{\tau^2}$: and the whole motive force constantly

$$\text{lost} = \left(g - \frac{\ddot{x}}{\tau^2} \right) \times B.$$

Now as the motive force constantly lost by B, must be equal to the motive force constantly retarding its descent, we shall have,

$$\frac{\ddot{x}}{y} \times \left(g + \frac{\ddot{y}}{\tau^2} \right) \times 2A = \left(g - \frac{\ddot{x}}{\tau^2} \right) \times B$$

where $y^2 = a^2 + x^2$.

Now $y\dot{y} = x\dot{x}$: therefore $\frac{\ddot{x}}{y} = \frac{\dot{y}}{\dot{x}}$. Let $\frac{B}{2A} = m$; then by substitution and properly ordering the terms, we get,

$$\frac{m \times \ddot{x} + \dot{y}\dot{y}}{\tau^2} = g(m\dot{x} - \dot{y})$$

and taking the fluents, recollecting that τ is constant,

$$\frac{m\dot{x}^2 + \dot{y}^2}{2\tau^2} = g \times (mx - y) + d$$

where d is a constant quantity necessary to correct the fluent.

In the last equation substitute $\frac{x^2\ddot{x}}{y^2}$ for its equal $\dot{y}^2$, and also v for $\frac{\dot{x}}{\tau}$, which is the measure of the velocity of B, corresponding to $2d$ or x , and we shall obtain

$$v = y \times \frac{\sqrt{2g(mx - y) + 2d}}{\sqrt{mx^2 + y^2}}$$

which corresponds exactly with the expression for the velocity obtained by Simpson.

Therefore, by the solution just given, the accuracy of Simpson's conclusion is confirmed.

The force gL opposes the force GH , and their difference $\{ = (w - q) \sin. \phi \}$ only acts on the cylinder, and is employed in urging it to slip off the support A . No part of this force can be destroyed by the force AN acting at right angles to it. We may suppose it to be counter-balanced by a weight $P = (w - q) \sin. \phi$ acting in the direction of the axis of the cylinder by means of a thread passing over a pulley.

The remaining forces gM , GK , and AN , are now all parallel, and may be in equilibrio. The conditions are, first, that the sum of the forces gM and AN be equal to the force GK ; and, secondly, that the force gM be to the force AN as GA to Gg .

Thus we have

$$GA : Gg :: q \cos. \phi : (w - q) \sin. \phi$$

whence

$$GA : gA :: q : w.$$

The weight $P = (w - q) \sin. \phi$, becomes smaller and smaller as the angle ϕ becomes less and less: and the angle ϕ may be so very small that the friction at A , together with the resistance of the fluid, may be a sufficient counterpoise to the tendency to slip. This must be supposed to be the case in the 10th question of the Gentleman's Diary for 1789; for otherwise the equilibrium supposed in the question could not subsist. Taking the numbers in that question; $AD = 36$ inches, the diameter of

the cylinder $= \frac{3}{4}$ of an inch, AC reckoned on the axis $= 13$ inches, $AB = 3$ inches and the weight of a cubic inch of the fluid $= 0.5949$ oz. avoirdupois. Hence $q = \frac{9}{16} \times 0.7854$

$\times 23 \times 0.5949 = 6.045$ oz. nearly; and $Ag = 24\frac{1}{2}$; $AG = 18$: therefore $18 : 24\frac{1}{2} :: 6.045 : 8.228 = w =$ weight of the cylinder required.

The angle ϕ will be determined from this equation $3 = 13 \sin. \phi - \frac{3}{8} \cos. \phi$; whence $\sin. \phi = 0.2585$ and $\cos. \phi = 0.9660$:

therefore the force AN , or the pressure at $A = (w - q) \cos. \phi = 2.099$ oz. and the weight P , equivalent to the tendency to slip off the support A , $= (w - q) \sin. \phi = 0.564$ oz. or something more than half an ounce.

2. If it be required to have regard to the dimensions of the cylinder, in the solution of such problems as the foregoing, it will become necessary to determine the centre of gravity of the part of a right cylinder cut off by a plane oblique to the axis.

For this purpose let it be required to determine the centre of gravity of the cylindrical hoof, ABC (fig. 139, pl. 8.) Let the oblique

oblique elliptical surface of the hoof touch the circular base in A, and let ACB be a section by a plane passing through A and the axis of the cylinder: also let DEGF be a section by a plane perpendicular both to the base and to ACB: then it is easy to prove that the section EF is a rectangle, that $DH = HE$ and $FK = KG$, and that, having drawn KH, it is parallel to BC and = FD or GE. Bisect BC in L, and draw AL meeting KH in M: then, it is obvious, that $KM = MH$, and that the point M is the centre of gravity of the rectangular section EF. Thus, the centres of gravity of all the sections of the hoof by planes parallel to EF are in the line AL: therefore the centre of gravity of the hoof itself will be in the line AL.

Let S represent the superficial content of the section of a solid homogeneous body by a plane, and x the distance of the section from a given parallel plane; then $\int Sx^2$ (taken between the limits of the greatest and least values of x , that is, from the value of x where S is nearest the given plane to the value of x where x is most remote from that plane) = the solid content of the body multiplied by the distance of the centre of gravity from the given plane. To apply this formula to the case in hand, put ϕ = measure of the arch BD, $BC = h$, and r = radius of the circular base: then $DE = 2r \sin. \phi$, DF

$$= KH = \frac{BC}{BA} \times AH = \frac{h}{2r} \times r (1 + \cos. \phi) = \frac{h}{2} (1 + \cos. \phi);$$

therefore $S = hr (1 + \cos. \phi) \sin. \phi$: also $x = r (1 - \cos. \phi)$: therefore $Sx^2 = hr^3 \times \phi \sin.^2 \phi (1 + \cos. \phi) (1 - \cos. \phi) =$
 $hr^3 \times \phi \sin.^4 \phi = hr^3 \times \phi \left\{ \frac{3}{8} - \frac{1}{2} \cos. 2 \phi + \frac{1}{8} \cos. 4 \phi \right\}$:

$$\text{therefore } \int Sx^2 = hr^3 \times \left\{ \frac{3}{8} \phi - \frac{1}{4} \sin. 2 \phi + \frac{1}{32} \sin. 4 \phi \right\}:$$

taking the whole fluent from $x = 0$ to $x = 2r$, or from $\phi = 0$ to $\phi = \text{semi-circumference} = \pi$, we have $\int Sx^2 = hr^3 \times \frac{3\pi}{8}$.

Now the solidity of the hoof = a cylinder on the same base and half the height $= hr^2 \times \frac{\pi}{2}$. Therefore the distance of the centre of gravity of the hoof from a plane touching the cylindrical surface in BC $= \frac{hr^3 \times \frac{3}{8} \pi}{hr^2 \times \frac{1}{2} \pi} = \frac{3}{4} r$. Hence, if o be the cen-

tre of the base, take $om = \frac{1}{4} r$, and draw mn parallel to BC, meeting AL in n , and n will be the centre of gravity of the hoof required.

It is now easy to find the centre of gravity of a portion of a right cylinder cut off by a plane oblique to the axis. Let $PwoQ$ be the axis, r the middle of oQ , or the centre of gravity of the cylinder AN ; join nr , and make nT to Tr , as the cylinder AN to the hoof ABC , that is, as OL to OP , and T will be the centre of gravity of the whole portion of the cylinder cut off by the oblique plane $AGFC$.

Draw ns and Tq perpendicular to PQ : Then $sw = \frac{1}{4}wo =$

$\frac{1}{8}Po$ and $rw = \frac{1}{2}PQ$; therefore $rs = \frac{1}{2}PQ + \frac{1}{8}PO$:

but $Po : QP :: rT : nr :: qr : rs$;

therefore $rq = \frac{1}{2}Po + \frac{1}{8} \cdot \frac{Po^2}{PQ}$. Hence

$$Qq = \frac{1}{2}PQ + \frac{1}{8} \cdot \frac{Po^2}{PQ}$$

$$Tq = \frac{1}{4} \cdot \frac{AO \cdot OP}{PQ}.$$

3. Now let it be required to resolve the problem considered in § 1, taking into account the dimensions of the cylinder. Let G (fig. 140, pl. 8.) be the middle of the axis AD , or the centre of gravity of the whole cylinder, and s the middle of the part CD : put $AD = a$, $As = b$, $2r =$ diameter of the cylinder, and $\phi =$ angle $ACB =$ inclination of the cylinder to the surface of the fluid: then take $sr = \frac{r^2 \cot.^2 \phi}{16(a-b)}$ and $rg = \frac{r^2 \cot. \phi}{8(a-b)}$, and

g will be the centre of gravity of the immersed part of the cylinder (§ 2). Then proceeding as in § 1, the force of gravity GE ($= w =$ weight of the cylinder) will be resolved into two forces, the one $GK = w \cos. \phi$, acting at right angles to the axis of the cylinder; and the other $GH = w \sin. \phi$, acting parallel to the axis: and in like manner, the force of the fluid on the immersed part of the cylinder, gF ($= q =$ weight of a portion of fluid equal in bulk to the immersed part of the cylinder) will be equivalent to two other forces, one $gM = q \cos. \phi$, perpendicular to AD , and one $gL = q \sin. \phi$, parallel to DA .

The forces GH and gL oppose one another: join Gg , and produce it p , making $w - q : q :: Gg : Gp$, and draw pn parallel to the axis of the cylinder: then the two opposing forces GH and gL will be equivalent to a single force applied to p or n and $= (w - q \sin. \phi)$, the difference of the forces GM and gL . No part of this force can be destroyed by the pressure on the obstacle (friction being excluded) which reacts on the cylinder in a direction

a direction perpendicular to the axis. We may suppose it to be counter-balanced by a weight $P = (w - q) \sin. \phi$, acting in the direction pn by means of a thread, fastened to the point n , passing over a pulley. The distance of n from the centre of the

$$\text{cylinder, or } nA = \frac{r^2 \cot. \phi}{8(a-b)} \times \frac{q}{w-q}.$$

From the conditions necessary to the equilibrium of the other three forces, we easily deduce, as in § 1,

$$GA : rA :: q : w$$

or in symbols

$$\frac{1}{2} a : b - \frac{r^2 \cot.^2 \phi}{16(a-b)} :: q : w$$

and the pressure on the obstacle $= (w - q) \cot. \phi$.

Taking the numbers in § 1, we have $q = 6.045$ oz: $a = 36$; $b = 24\frac{1}{2}$: and $\phi = 14^\circ 59'$: and $r = \frac{3}{8}$: whence

$18 : 24.489 :: 6.045 : 8.224$ oz. $= w$ the weight of the cylinder required.

4. Suppose now that AD (fig. 141, pl. 8.) is a long cylindrical beam of timber, moveable at one end round an axis (passing through the centre of the cylinder at right angles to its axis) supported by a prop AB of a given height above the surface of a fluid in which the beam floats: it is required to determine the position in which the beam will rest in equilibrio.

Let c = height of the centre of the axis, round which the beam turns, above the surface of the fluid; a = length of the cylinder, and ϕ = inclination to the surface of the fluid: then

$\frac{c}{\sin. \phi} =$ length of the part of the axis of the cylinder out of

the fluid, and $(a - \frac{c}{\sin. \phi}) =$ length of the immerfed part: let

$w =$ weight of the cylinder, $w' =$ weight of a portion of fluid

equal in bulk to the cylinder, then $(1 - \frac{c}{a \sin. \phi}) \times w' =$

weight of a portion of fluid equal in bulk to the immerfed part

of the cylinder. Take $Dr = \frac{1}{2} (a - \frac{c}{\sin. \phi}) + \frac{1}{8} \cdot \frac{r^2 \cot.^2 \phi}{a - \frac{c}{\sin. \phi}}$

($2r$ being the diameter of the cylinder) and $gr = \frac{1}{4} \cdot \frac{r^2 \cot.^2 \phi}{a - \frac{c}{\sin. \phi}}$;
then

then will g be the centre of gravity of the immersed part of the cylinder: draw the vertical line gF meeting the axis AD in m ;

then $rm = gr \times \tan. \phi = \frac{1}{4} \cdot \frac{r^2}{a - \frac{c}{\sin. \phi}}$: hence $Am = AD$,

$$Dr - rm = \frac{1}{2} \left(a + \frac{c}{\sin. \phi} \right) - \frac{1}{8} \cdot \frac{2r^2 + r^2 \cot. \phi}{a - \frac{c}{\sin. \phi}}.$$

The only forces that act on the beam are the force of gravity $= w$ (the weight of the beam) in the direction GE , and the force $gF = \left(1 - \frac{c}{a \sin. \phi} \right) \times w'$, and the beam will be in equilibrio in that position where these two forces are in equilibrio, that is, when

$$AG : Am :: \text{force } gF : \text{force } GE$$

or, in symbols,

$$\frac{a}{2} : \frac{1}{2} \left(a + \frac{c}{\sin. \phi} \right) - \frac{1}{8} \cdot \frac{2r^2 + r^2 \cot. \phi}{a - \frac{c}{\sin. \phi}} :: \left(1 - \frac{c}{a \sin. \phi} \right) w' : w.$$

$$\text{Hence } a^2 - \frac{c^2}{\sin.^2 \phi} - \frac{r^2}{4} \times (2 + \cot.^2 \phi) = a^2 \cdot \frac{w}{w'}.$$

$$\text{and } \sin. \phi = \sqrt{\frac{c^2 + \frac{r^2}{4}}{a^2 \left(1 - \frac{w}{w'} \right) - \frac{r^2}{4}}}$$

If in the formula just found, we make $c^2 + \frac{r^2}{4} = a^2 \left(1 - \frac{w}{w'} \right) - \frac{r^2}{4}$ or $c = \sqrt{\left\{ a^2 \left(1 - \frac{w}{w'} \right) - \frac{r^2}{2} \right\}}$, then $\sin. \phi = 1$ and

$\phi = 90^\circ$: therefore when c has this value, and much more when c has a greater value (in which case the formula gives the value of $\sin. \phi$ greater than the radius) the beam will not rest in equilibrio in any position oblique to the horizon: but, when c has a less value, there will be one position, oblique to the horizon, as determined by the formula, in which the beam will rest in equilibrio.

Let it be remarked, however, that the formula is limited in its application; for it will not apply when the prop is so short, that either the lower edge of the upper end of the cylinder is immersed in the fluid, or the upper edge of the lower end of the cylinder rises out of the fluid. To investigate these cases it would be necessary

necessary to determine the centre of gravity of a hoof of a right cylinder insisting on a given part, or segment of the base.

5. Suppose now that the long and slender cylinder (fig. 142, pl. 8.) AD, floating in a fluid, is kept in equilibrio by a weight P, acting on the cylinder by means of a thread ANP passing over a pulley, and fastened to the extremity of the axis of the cylinder: and let it be required to determine the weight P when the axis of the cylinder has a given inclination to the surface of the fluid.

Although this problem may be easily deduced from the problem last considered, yet we shall give an independant solution of it.

Let G be the middle of the axis, or the centre of gravity of the cylinder; and S the middle of the immerfed part of the axis;

put $AD = a$, $SD = x$, and take $Dr = x + \frac{1}{8} \cdot \frac{r^2 \cot.^2 \varphi}{2x}$ (φ

being the given inclination of the axis to the surface of the fluid,

and $2r$ the diameter of the cylinder) and $rg = \frac{1}{4} \cdot \frac{r^2 \cot.^2 \varphi}{2x}$: then

g will be the centre of gravity of the immerfed part of the cylinder; draw the vertical line gF cutting the axis in m , then rm

$= gr \cdot \tan. \varphi = \frac{1}{4} \cdot \frac{r^2}{2x}$: therefore $Am = a - x - \frac{1}{8} \cdot \frac{r^2 \cot.^2 \varphi}{2x}$

$- \frac{1}{4} \cdot \frac{r^2}{2x}$. Also if w = weight of the cylinder, and w' = weight of

a portion of fluid equal in bulk to the cylinder, then $\frac{2x}{a} \times w'$

= weight of a portion of fluid equal in bulk to the immerfed part of the cylinder.

The forces that act on the cylinder are the two vertical forces, applied to the points m and A , equal to $\frac{2x}{a} \times w'$ and the weight P ; and the force in the direction of gravity, applied to G , and equal to w .

The conditions of equilibrium are first that $w = \frac{2x}{a} \times w' +$

P : and, secondly, that $AG : Gm :: \frac{2x}{a} \times w' : P$ or $w -$

$\frac{2x}{a} w' : \text{whence } AG : Am :: \frac{2x}{a} \times w' : w$, and $AG \times w =$

$Am \times \frac{2x}{a} \times w'$, or in symbols:

(a —

$$\left(a - x - \frac{1}{8} \cdot \frac{r^2 \cot.^2 \phi}{2x} - \frac{1}{4} \cdot \frac{r^2}{2x}\right) \times \frac{2x}{a} \times w' = \frac{a}{2} \times w:$$

$$\text{Thus } 4ax - 4x^2 = a^2 \cdot \frac{w}{w'} + \frac{1}{2} r^2 + \frac{1}{4} r^2 \cot.^2 \phi$$

$$\text{therefore } a - 2x, \text{ or } AC = \sqrt{\left\{a^2 \left(1 - \frac{w}{w'}\right) - \frac{1}{2} r^2 - \frac{1}{4} r^2 \cot.^2 \phi\right\}}$$

and x being thus known, the value of $P = w - \frac{2x}{a} \times w'$ will

likewise be known.

But it must be remarked, that this formula will not apply when the cylinder is so obliquely inclined to the surface of the fluid, that either the lower edge of the upper end of the cylinder is immersed in the fluid, or the upper edge of the lower end rises out of the fluid.

If in the formula that has been found we put $r = 0$, then

$$AC = a \sqrt{\left(1 - \frac{w}{w'}\right)}, \text{ which agrees with the result of Mr.}$$

Farey's investigation, (page 161, No. 14, of the Old Series of the Repository).

Taking the numbers in the experiment made by Mr. Farey, and of which some account is given in the Repository as above cited, we have $a = 28.12$ inches, $2r = 1.7$ inches, and $w = 17.65$ ounces: the experimenter informs us, that when the cylinder was placed upright and allowed to sink to its equilibrium in the water, he observed 13.92 inches of the axis to be immersed; but that when the same thing was done, inverting the cylinder, it sunk 14.45 inches into the water: the former of these

trials gives $w' = \frac{28.12}{13.98} \times 17.65$ oz. and the latter gives $w =$

$\frac{28.12}{14.45} \times 17.65$ oz.; and it appears reasonable to prefer a mean of

the two trials, and thus to take $w' = \frac{28.12}{14.215} \times 17.65$ or $\frac{w}{w'} =$

$\frac{14.215}{28.12}$. If these numbers be substituted in the formula we

shall find $AC = \sqrt{(390.648 - 0.108 \cot.^2 \phi)} =$ (by throwing the root into a series, and rejecting all the terms but the two first on account of their smallness,) $19.76 - 0.0045 \times \cot.^2 \phi$. The Author of the experiment informs us, that he observed 19.5 inches of the cylinder's axis to be above the water at several inclinations; and the theory, which gives 19.76 inches instead of 19.5 agrees as well with the fact as can reasonably be expected, considering

considering the uncertainty in the determination of w' . The correction depending on the inclination of the axis of the cylinder is manifestly too small to be of any account in the present instance.

We have CD , or $2x$, $= 8.36 \text{ in} + .0045 \cot.^2 \phi$: therefore $\frac{2x}{a} w' = \frac{17.65}{14.215} + \{ 8.36 + .0045 \cot.^2 \phi \} = 10.29 \text{ oz.} + .0055 \cot.^2 \phi$. In Mr. Farey's experiment the thread AN , fastened to the end of the cylinder, was attached to one scale of a balance, the weight P being put into the opposite scale: and we are informed that, while the balance was raised and depressed by means of a pulley, so as to make the cylinder pass into all degrees of obliquity to the surface of the fluid, the scales remained *as nearly as possible* in equilibrium, and the thread vertical, the counterpoise, or weight P , being $= 7.175 \text{ oz.}$: neglecting the correction depending on the inclination, the theory gives 7.36 instead of 7.175 : but when $\phi = 6^\circ$, then $\cot. \phi = 9\frac{1}{2}$ nearly, and the correction depending on the inclination $=$ half an ounce nearly: thus according to theory the counterpoise, or weight P , ought to diminish from 17.36 oz. to 6.86 oz. , while the cylinder passes from being perpendicular to the horizon to an inclination of 6° . The theory seems to agree with Mr. Farey's experiment as nearly as his observations afford us the means of judging. When the inclination of the axis, in this experiment, is between 5° and 6° , the upper edge of the lower end of the cylinder would, according to theory, just come to the surface of the water, and after this the formula we have given would no longer apply: but the great length to which this article has already run, renders it improper to add any thing more on this subject at present.

ARTICLE VIII.

Of the length of an arc of a circle in terms of the tangent.

By Mr. BENJAMIN GOMPERTZ.

LET w and v be the arcs corresponding to the tangents z and t respectively, radius being unity.

Assume

$$w = az + bz^3 + cz^5 + dz^7 + \&c.$$

$$\text{and } v = at + bt^3 + ct^5 + dt^7 + \&c.$$

$a, b, c, d, \&c.$ being constant quantities to be determined.

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Suppose

Suppose $w = 2v$, consequently,
 $az + bz^2 + c^2z + dz^4 + \&c. = 2at + 2bt^2 + 2ct^3 + 2dt^4 + \&c.$
 and by trigonometry,

$$z = \frac{2t}{1-t^2} = 2t + 2t^3 + 2t^5 + 2t^7 + \&c.$$

$$z^2 = \frac{4t^2}{(1-t^2)^2} = 4t^2 + 8t^4 + 12t^6 + \&c.$$

$$z^3 = \frac{8t^3}{(1-t^2)^3} = 8t^3 + 24t^5 + \&c. \&c. \&c.$$

Consequently writing these values of $z, z^2, z^3, \&c.$ in the above equation, and transposing the terms we get

$$\left. \begin{array}{r} 2at + 2bt^2 + 2ct^3 + 2dt^4 + 2et^5 \\ - 2at \dots \dots \dots - 2at^3 \dots \dots \dots - 2at^5 \\ \quad - 4bt^2 \dots \dots \dots - 8bt^4 \dots \dots \dots \\ \quad \quad - 8ct^3 \dots \dots \dots - 24ct^5 \\ \quad \quad \quad - 16dt^4 \dots \dots \dots \\ \quad \quad \quad \quad - 32et^5 \end{array} \right\} = 0,$$

Hence, comparing the homologous terms we have $2a = 2a$;
 $2b - 4b = 0$, therefore $b = 0$; $2c - 2a - 8c = 0$ or $-2a - 6c$
 $= 0$, therefore $c = -\frac{1}{3}a$; $2d - 8b - 16d = 0$, consequently,
 since $b = 0$, d is 0 ; $2e - 2a - 24c - 32e = 0$, that is, be-
 cause $c = -\frac{1}{3}a$, $e = \frac{1}{5}a$; and the like for the rest of the terms.

Consequently the length of an arc of a circle whose tangent is t
 and radius 1 is

$$at - \frac{at^3}{3} + \frac{at^5}{5} \&c. \text{ or } a \times \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \&c. \right)$$

which is general for any tangent whatever.

Now, in this expression, it is evident that the tangent may be
 taken so small that its ratio to the arc may differ from the ratio
 of 1 to a by less than any assignable quantity; and the tangent may
 also be taken so small, that its ratio to the arc may differ from the
 ratio of equality by less than any assignable quantity; therefore
 it follows that the ratio of 1 to a must be the ratio of equality, or

$$1 = a; \text{ and therefore the arc } = t - \frac{t^3}{3} + \frac{t^5}{5} - \&c.$$

This method of finding the length of a circular arc in terms of
 the tangent, occurred to me some years ago on reading a paper of
 De Moivre in the Philosophical Transactions, No. 240, and
 since then I find the same has been effected by L'eglange in his
 Théorie de Fonctions Analytique, by assuming the equation ax
 $+ \sin. hx \sqrt{-1} = (\cos. x + \sin. x \sqrt{-1})^n$.

ARTICLE IX.

GEOMETRICAL PORISMS.

By Mr. MARK NOBLE.

THE following propositions, though not of difficult investigation, serve to illustrate what I conceive to be a new method of conducting the geometrical analysis of Porisms, and which I discovered towards the close of the autumn of the year 1804.

PROPOSITION I. PORISM. Fig. 143. Pl. 8.

A plane triangle ABC being given in position, there is *given* a point L such that any straight line EF being drawn through L and CF, BG, AE, drawn parallel to one another, A, B, being always on the same side of EF, CF shall always have to AE + BG a ratio the same as a given ratio, M to N.

ANALYSIS. Suppose the porism true, then if CL be joined, meeting AB in H, CH and H will be *given**; draw HK parallel to BG, then $HK : FC :: HL : LC$, that is, the ratio of HK to FC is *given*, but the ratio of FC to AE + BG, is given, therefore (8 dat.) the ratio of HK to AE + BG is *given*, but if FE meet AB in P, $HK : AE + BG :: PH : BP + PA$, therefore the ratio of PH to BP + PA is *given*; therefore, invertendo, the ratio of BP + PA to PH is *given*, and, dividendo, BH + PA has a *given* ratio to PH or PA + AH; now this is either a ratio of equality, or one of the terms is greater than the other; let which you will as BH + PA be the greater; then, dividendo, the ratio of BH - AH to PA + AH is *given*, but BH, AH, are *given*, therefore AP is *given* in magnitude; therefore P is *given*, which is absurd; therefore BH + AP = PA + AH and BH = AH; therefore H is given, and HC is given in position. And since AH = HB, $HK : AE + BG :: 1 : 2$, and the ratio of AE + BG : FC :: N : M, therefore $HK : FC :: N : 2M$; but $HK : FC :: HL : LC$; therefore $HL : LC :: N : 2M$; therefore L is given Q. E. I.

* In these investigations, when the word given is used to denote any thing which the proposition affirms to be given, and which is to be found, it is printed in *italics*.

PROPOSITION II. PORISM. Fig. 144, Pl. 8.

Two straight lines, AC, CB, being given in position, and therein two points, A, B, there is *given* a point D, such, that any circle thence described shall cut off segments AE, BF, adjacent to the given points, having between themselves a *given* ratio.

ANALYSIS. Suppose it true : and let D be the point which is affirmed to be *given*, and let the circle meet the lines again in K, L ; then, by hypothesis, the ratio of AK to BL is *given*, and the same with that of AE to BF, therefore the ratio of EAK to FBL is *given*, that is, the ratio of $AD^2 - DG^2$ to $BD^2 - DH^2$ is given ; also $DG = DH$, and AD, DB, are *given* ; now since the aforesaid ratio is given it is either of equality or not ;—if not, let which of the terms you will as $AD^2 - DG^2$ be the greater, then, dividendo $AD^2 - BD^2$ has a *given* ratio to $BD^2 - DH^2$, therefore since AD, BD, are *given*, DH is also *given*, contrary to the hypothesis ; therefore the ratio is of equality, and $AD = BD$; and therefore, since $AE = BF$ and $AK = BL$, $EK = FL$, that is, they are equidistant from D ; whence D is given, being the intersection of DO, which bisects and is perpendicular to AB, and CD which bisects the angle ACL ; and it is evident that D may be found in either of the four angles at C.

PROPOSITION III. PORISM.

If there be any given number of points, and a circle given by position ; putting n for the given number, if a polygon of $n+1$ sides be inscribed in the circle so that a side may pass through each of the given points, the remaining side shall contain a *given* angle at the circumference with a straight line tending to a *given* point.

CASE I. If in the circle CDE, given in position, the triangle DCE be inscribed so that DC, CE, tend to A, B, given points, ED makes a *given* angle at D, with a line that tends to a *given* point. Fig. 145, pl. 8.

ANALYSIS. Suppose it true, and let F be the point which in the porism is affirmed to be *given*, then if DF be drawn meeting the circle in G, the angle GDE is *given* ; join AB, parallel to which draw EH and join DH, meeting AB in K, then the angle $DCE = KHE = DKA$, therefore $KA \times AB = DA \times AC$; therefore K is a given point. Join GH, then the angle $GHE = EDF$ is *given*, but HE is parallel to a straight line AB

iven

given in position; therefore HG is also parallel to a straight line given in position; join KF meeting the circle in T, V; then, since K is given and F given, KV is given in position, parallel thereto draw HX, then since HG and HX are parallel to lines given in position, the angle XHG is given, therefore GX is given in magnitude: also, since KDF is inscribed from the points F, K, and HX drawn parallel to FK, and GX joined meets FK in S, S may be shewn to be given in the same way that K was. Now, since through the given point S is drawn GX given in magnitude but not in position, S is the centre, and GX consequently a diameter, and XHG a right angle, therefore the angle FDE = GHE = difference of BKS = XHF and a right angle, but K and S are given, therefore the angle SKB is given, and therefore the angle FDE is given, and since $KF \times FS = TF \times FV^*$, F is given, viz. by making $TK : KV :: TF : FV$ Q. E. I.

This is the 57th prop. of Dr. Simson's treatise de porismatibus, and the last porism of the third book of Euclid's Porisms.

It may not be improper to enumerate its particular cases. The angle FDE, that is, the difference between the given angle SKB and a right angle may be either a right angle or nothing, that is, the angle SKB may be nothing (or SK coincide with AB, AB passing through the centre) or a right angle, in both cases K is found by making $KA \times AB = DA \times AC$.

When AB passes through the centre the angle FDE is a right angle, and here we have two particular cases.

First, if K and F are the given points, HG is perpendicular to the diameter, i. e. HG makes a right angle with HX, which tends to a point at an infinite distance. This case is prop. 58, de porismatibus.

Secondly, which is in fact the converse, if KHD be drawn and HG perpendicular to KS, then DG tends to a given point F: i. e. HG is drawn from a point infinitely distant in a line perpendicular to the diameter and passing through K; this is prop. 62, de porismatibus, case the fifth.

When AB does not pass through the centre, there are likewise two particular cases:

First, when SK is perpendicular to AB then the angle EDF is nothing, that is, ED passes through the given point; this is prop. 61, de porismatibus.

Secondly, when AB is given in position and from K, any point in it is drawn KD, and HE tending to a point infinitely distant in it, i. e. parallel to AB, DE still makes a given angle EDF, found as before, with a line tending to the given point F. This is the second case of prop. 62, de porismatibus.

The analysis of every one of these cases is contained in the

* By the similar triangles DFK, GFS.

general analysis, the general proposition having been reduced to them one after another; but I have been thus precise in the enumeration of them that the beautiful connection between the propositions cited, might be clearly seen. I may also observe, that this analysis is in every respect different from Dr. Simson's, except that both are grounded upon the principle of the analysis of prop. 117, lib. 7. Pappi, which Dr. Simson has made, prop. 55, de porismatibus.

LEMMA I. Fig. 146, Pl. 8.

If from a given point D, to a circle given in position, be any how drawn DC, and from C be drawn CA in a given angle DCA, and AB be drawn parallel to a straight line given in position, and BC be joined; BC makes at C a *given* angle with a straight line that tends to a *given* point.

DEMONSTRATION. Join BG, then the angle ABG is given, because ACD is given, therefore BG, making a given angle with a line that is parallel to a line given in position, is itself parallel to a line given in position, therefore the lemma is evident by that case of the preceding porism that makes prop. 62, de porismatibus.

LEMMA II. Fig. 147, Pl. 8.

If from two given points D, E, the straight lines DA, AE, be inflected to any point A in a given circumference meeting it again in C, F, and from A be drawn AB to the circumference in a given angle EAB, BC joined makes at C a *given* angle with a line tending to a *given* point.

DEMONSTRATION. Join FC, which, by the foregoing porism, makes a given angle with a line tending to a given point G, but the angle BCF (= EAB) is given, therefore the angle BCG is given.

CASE 2. When there are three or more points F, G, H, K, &c.

DEMONSTRATION. Having drawn the diagonals AC, AD, &c. fig. 148, plate 8. By the first case AC makes a *given* angle with a line that tends to a given point, therefore by the second lemma AD also makes a given angle with a line tending to a given point, and, therefore, by the same lemma, AE also makes a given angle with a line tending to a given point; and so on, let the number of given points be what it may. Therefore the proposition is evident.

COR. Hence it is evident how a polygon may be inscribed in a circle so that its sides may pass through any given number

of given points, or be parallel (lem. 1.) to any given number of lines given by position, or some pass through given points and some be parallel to, or even (lem. 2.) make given angles with lines tending to given points.

The following porism is added here from its similarity to the first case of the foregoing, and as a further specimen of my method of investigating this class of propositions.

PROPOSITION IV. PORISM. Fig. 149, Pl. 8.

A circle whose centre is A, and a straight line BC being given in position and magnitude, if through B, C, any circle be described cutting the given circle in D, E; DE joined tends to a given point.

ANALYSIS. Suppose the porism true, and F the point *given*, join FC meeting the circle again in G, then $CF \times FG (= EF \times FD)$ is *given*, because F is *given* and the circle HED *given*, but FC is *given* in magnitude and position, therefore FG is *given* in magnitude and position; therefore G is a *given* point; therefore G coincides with B, for otherwise the circle CBG would be given in position, which it is not; join FA and bisect BC in K, then since it has been proved above that $(KF)^2 - (KC)^2 = (AF)^2 - (AH)^2$, the difference of the squares of KF and AF, being equal to the difference of the squares of KC and AH, is given; therefore (1 prop. 2, lib. loc. plan.) F touches a straight line given in position, and it has been proved to be in BC produced if need be; therefore F is given Q. E. I.

And it appears from the proposition cited, that if AK joined is perpendicular to BC, F is at an infinite distance, i. e. DE is parallel to BC.

ARTICLE X.

DIOPHANTINE PROBLEMS.

By Mr. JAMES CUNLIFFE, R. M. College.

PROBLEM I.

TO find three square numbers such that the sum of every two of them may be a square number.

SOLUTION.

SOLUTION.

Let x^2 , y^2 , and z^2 , denote the numbers; then by the question $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$ must all be rational squares.

Assume $x + y = a$, for the root of the first, and $x + z = b$, for the root of the second, that is, put

$$x^2 + y^2 = (x + y - a)^2 = x^2 + y^2 + 2xy - 2a(x + y) + a^2, \text{ and}$$

$$x^2 + z^2 = (x + z - b)^2 = x^2 + z^2 + 2xz - 2b(x + z) + b^2;$$

from the former of which $x = \frac{a^2 - 2ay}{2(a - y)}$, and from the latter

$$x = \frac{b^2 - 2bz}{2(b - z)}; \text{ therefore } \frac{a^2 - 2ay}{2(a - y)} = \frac{b^2 - 2bz}{2(b - z)}; \text{ whence}$$

$$z = \frac{by(2a - b) - ba(a - b)}{2y(a - b) + a(2b - a)};$$

$$\text{put } a - b = d, \text{ then } z = \frac{by(2a - b) - ba(a - b)}{2y(a - b) + a(2b - a)} = \frac{(a^2 - d^2)y - ad(a - d)}{2dy + a^2 - 2ad}$$

$$\text{and hence } y^2 + z^2 = y^2 + \frac{\{(a^2 - d^2)y - ad(a - d)\}^2}{(2dy + a^2 - 2ad)^2}$$

$$= \frac{y^2(2dy + a^2 - 2ad)^2}{(2dy + a^2 - 2ad)^2} + \frac{\{(a^2 - d^2)y - ad(a - d)\}^2}{(2dy + a^2 - 2ad)^2}$$

which must be a square by the question, therefore

$$\left\{ \frac{y^2(2dy + a^2 - 2ad)^2}{(2dy + a^2 - 2ad)^2} + \frac{\{(a^2 - d^2)y - ad(a - d)\}^2}{(2dy + a^2 - 2ad)^2} \right\} = \left\{ 4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 + (a^2 - d^2)^2\} \right. \\ \left. - 2ady(a - d)(a^2 - d^2) + a^2d^2(a - d)^2 \right\} = \text{a square.}$$

Assume $2y^2d + y(a^2 - 2ad) - ad(a - d)$ for its root; then

$$4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 + (a^2 - d^2)^2\} \\ - 2yad(a - d)(a^2 - d^2) + a^2d^2(a - d)^2$$

$$= \{2y^2d + y(a^2 - 2ad) - ad(a - d)\}^2$$

$$= 4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 - 4ad^2(a - d) \\ - 2yad(a^2 - 2ad)(a - d) + a^2d^2(a - d)^2\}$$

$$\text{which being reduced gives } y = \frac{2ad(2ad - d^2)}{(a + d)^2(a - d) + 4ad^2}, \text{ and by}$$

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restoring the value of d , $y = \frac{2a(a-b)(a^2-b^2)}{4a^3+b^3-4a^2b}$, where a and b may be taken at pleasure (A). This I call the first formula for the value of y , and have marked it A.

Example 1. Suppose $a = 3$, and $b = 2$,

$$\text{then, } y = \frac{15}{22}; x = \frac{a^2 - 2ay}{2(a-y)} = \frac{9 - \frac{90}{22}}{2(3 - \frac{15}{22})} = \frac{198 - 90}{2(66 - 15)} = \frac{108}{102} = \frac{18}{17};$$

$$\text{and } z = \frac{b^2 - 2bx}{2(b-x)} = \frac{4 - \frac{72}{17}}{2(2 - \frac{18}{17})} = \frac{68 - 72}{2(34 - 18)} = \frac{-4}{32} = \frac{-1}{8}.$$

Therefore $\frac{15}{22}$, $\frac{18}{17}$ and $\frac{1}{8}$ are the roots of three squares, the

sum of every two of which will be a rational square. If whole numbers are desired they may be obtained by reducing the fractions to a common denominator, and rejecting it; when the preceding fractions are so ordered there will be had $15 \times 17 \times 8 = 2040$; $18 \times 22 \times 8 = 3168$; and $22 \times 17 = 374$ for the roots of three squares that will answer; or because these numbers are all divisible by 2; 1020, 1584, and 187 are the roots of three squares that will answer.

Example 2. Take $a = 1$, and $b = -2$; then $y = \frac{-9}{2}$;

$$x = \frac{a^2 - 2ay}{2(a-y)} = \frac{10}{11}; \text{ and } z = \frac{b^2 - 2bx}{2(b-x)} = \frac{-21}{16}; \text{ from whence,}$$

by reducing the fractions to a common denominator, we shall obtain 792, 231, and 160 for the roots of three whole numbers, the sum of the squares of every two of which will be a square.

A second formula for the value of y may be obtained as follows.

Let $2y^2d + y(a^2 - 2ad) + ad(a - d)$ be assumed for the root of the general expression to be made a square; that is, put

$$\begin{aligned} & 4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 + (a^2 - d^2)^2\} \\ & \quad - 2yad(a - d)(a^2 - d^2) + a^2d^2(a - d)^2 \\ & = \{2y^2d + y(a^2 - 2ad) + ad(a - d)\}^2 \\ & = 4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 + 4ad^2(a - d)\} \\ & \quad + 2yad(a - d)(a^2 - 2ad) + a^2d^2(a - d)^2 \end{aligned}$$

$$\text{which being properly reduced gives } y = \frac{2ad(2a^2 - 2ad - d^2)}{(a + d)^2(a - d) - 4ad^2};$$

and by restoring the value of d , $y = \frac{2a(a-b)\{4ab-(a^2+b^2)\}}{12a^2b+b^3-4a^3-8ab^2}$, (B)

which is a second formula, for the value of y , where a and b may, as before, be taken at pleasure.

Example i. Take $a = 2$ and $b = 1$, then $y = 12$; $x = \frac{a^2 - 2ay}{2(a-y)} = \frac{11}{5}$; and $z = \frac{b^2 - 2bx}{2(b-x)} = \frac{17}{12}$, which are the roots of three squares, the sum of every two of which will be a square; and these being ordered, as in the examples to formula A, will give $12 \times 5 \times 12 = 720$, $11 \times 12 = 132$, and $17 \times 5 = 85$ for the roots of three whole numbers that will answer.

Example 2. Take $a = 1$ and $b = -1$; then $y = \frac{24}{25}$; $x = \frac{a^2 - 2ay}{2(a-y)} = \frac{-23}{2}$; and $z = \frac{b^2 - 2bx}{2(b-x)} = \frac{-22}{21}$; and by ordering as before we shall have $24 \times 2 \times 21 = 1008$, $23 \times 25 \times 21 = 12075$, and $22 \times 25 \times 2 = 1100$ for the roots of three other whole numbers that will answer.

Again, assume $2y^2d - y(a^2 - d^2) + ad(a - d)$ for the root of the expression which is to be made a square, that is, make

$$\begin{aligned} & 4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 + (a^2 - d^2)^2\} \\ & \quad - 2yad(a - d)(a^2 - d^2) + a^2d^2(a - d)^2 \\ & = \{2y^2d - y(a^2 - d^2) + ad(a - d)\}^2 \\ & = 4y^4d^2 - 4y^3d(a^2 - d^2) + y^2\{(a^2 - d^2)^2 + 4ad^2(a - d)\} \\ & \quad - 2yad(a - d)(a^2 - d^2) + a^2d^2(a - d)^2, \end{aligned}$$

from whence, by proper reduction, $y = \frac{4ad^2(a - d) - (a^2 - 2ad)^2}{4d(2a^2 - 2ad - d^2)}$,

and restoring d , $y = \frac{a(8a^2b + 4b^3 - 12ab^2 - a^3)}{4(a-b)\{4ab - (a^2 + b^2)\}}$ (C) which

is a third formula for y .

Example 1. Take $b = 2$ and $a = 1$, then $y = \frac{1}{12}$, $x = \frac{a^2 - 2ay}{2(a-y)} = \frac{5}{11}$, and $z = \frac{b^2 - 2bx}{2(b-x)} = \frac{12}{17}$; and these being reduced to a common denominator as in the foregoing examples, and that denominator rejected, we shall have $1 \times 11 \times 17 = 187$,

$= 187, 5 \times 12 \times 17 = 1020$, and $12 \times 12 \times 11 = 1584$ for the roots of three whole numbers that will answer, and are the same as were found in one of the preceding examples.

Example 2. Take $a = 4$, and $b = 1$; then $y = \frac{-20}{3}$,

$$x = \frac{a^2 - 2ay}{2(a-y)} = \frac{13}{4}; \text{ and } z = \frac{b^2 - 2bx}{2(b-x)} = \frac{11}{9}; \text{ these be-}$$

ing reduced to the least common denominator and that denominator rejected, there will be had 240, 117, and 44 for the roots of three whole numbers that will answer.

Lastly, assume $2y^2d + y(a^2 - d^2) - ad(a - d)$ for the root of the general expression to be made a square, or put

$$4y^4d^2 + 4y^3d(a^2 - 2ad) + y^2\{(a^2 - 2ad)^2 + (a^2 - d^2)^2\} \\ - 2yad(a - d)(a^2 - d^2) + a^2d^2(a - d)^2$$

$$= \{(2y^2d + y(a^2 - d^2) - ad(a - d))\}^2$$

$$= 4y^4d^2 + 4y^3d(a^2 - d^2) + y^2\{(a^2 - d^2)^2 - 4ad^2(a - d)\} \\ - 2yad(a - d)(a^2 - d^2) + a^2d^2(a - d)^2$$

$$\text{and this after proper reduction gives } y = \frac{(a^2 - 2ad)^2 + 4ad^2(a - d)}{4d^2(2a - d)},$$

and restoring the value of d , $y = \frac{a(a^3 + 4b^3 - 4ab^2)}{4(a - b)(a^2 - b^2)}$ (D) and this is a fourth formula for y .

Example 1. Take $a = 2$ and $b = -1$; then $y = \frac{-2}{9}$,

$$x = \frac{a^2 - 2ay}{2(a-y)} = \frac{11}{10} \text{ and } z = \frac{b^2 - 2bx}{2(b-x)} = \frac{-16}{21}; \text{ from whence,}$$

by proceeding as in the former examples, we shall obtain 140, 693, and 480 for the roots of three squares that will answer.

Example 2. Take $a = 1$, and $b = 2$; then $y = \frac{17}{12}$,

$$x = \frac{a^2 - 2ay}{2(a-y)} = \frac{11}{5}, \text{ and } z = \frac{b^2 - 2bx}{2(b-x)} = 12, \text{ and from hence}$$

we shall obtain 85, 132, and 720 for the roots of three squares that will answer.

Example 3. Take $a = 3$, and $b = 4$; then $y = \frac{39}{4}$,

$$x = \frac{a^2 - 2ay}{2(a-y)} = \frac{11}{3}, \text{ and } z = \frac{b^2 - 2bx}{2(b-x)} = -20, \text{ from whence}$$

we

we shall obtain 44, 117, and 240 for the roots of three numbers that will answer.

Example 4. Take $a = 3$, and $b = 1$, then $y = \frac{57}{64}$, $x = \frac{13}{15}$, and $z = \frac{11}{4}$, and from these we get 832, 855, and 2640 for the roots of three numbers that will answer.

There is already a pretty general answer to the preceding question in No. 12 of the Old Series of the Repository; but the solution here given is far more comprehensive, as will appear from the great variety of examples which are annexed to illustrate the different formulæ. I believe it would be difficult to obtain some of the sets of numbers which are deduced in the annexed examples by any other method of solution.

PROBLEM II.

To find values for the sides of a triangle in whole numbers such that the lengths of the three lines from the angles to the middle of the opposite sides may be expressed by rational whole numbers.

SOLUTION.

Let ACB (fig. 150, pl. 7.) be a triangle, AE, BF, and CD lines from the angles to the middle of the opposite sides. Put $AF = FC = x$, $BE = EC = y$, and $AD = DB = z$. Then by prop. 28, B. 2d. Emerson's Geometry,

$(AC)^2 + (BC)^2 = 2(AD)^2 + 2(CD)^2$; whence

$$\frac{(AC)^2 + (BC)^2}{2} - (AD)^2 = 2x^2 + 2y^2 - z^2 = (CD)^2 = \text{a square,}$$

and in like manner $2x^2 + 2z^2 - y^2 = (AE)^2 = \text{a square;}$

and $2y^2 + 2z^2 - x^2 = (BF)^2 = \text{a square.}$

Put $x - y + z = n$, then

$$\begin{aligned} 2x^2 + 2y^2 - z^2 &= 2x^2 + 2y^2 - (x - y + n)^2 \\ &= x^2 + 2xy + y^2 - 2n(x - y) - n^2 \end{aligned}$$

= a square. Assume $x + y - s$ for its root, that is, put

$$\begin{aligned} x^2 + 2xy + y^2 - 2n(x - y) - n^2 &= (x + y - s)^2 \\ &= x^2 + 2xy + y^2 - 2s(x + y) + s^2; \end{aligned}$$

from

from whence $x = \frac{2y(n+s) - (n^2 + s^2)}{2(n-s)}$; and

$$z = x - y + n = \frac{2y(n+s) - (n^2 + s^2)}{2(n-s)} - y + n = \frac{4sv + n^2 - 2sn - s^2}{2(n-s)}.$$

From what has been deduced it is manifest that we may take $x = 2v(n+s) - (n^2 + s^2)$, $y = 2v(n-s)$ & $z = 4sv + n^2 - 2sn - s^2$, and by means of these

$$\begin{aligned} 2x^2 + 2y^2 - z^2 &= \left\{ 2 \left\{ 2v(n+s) - n^2 + s^2 \right\}^2 + 8v^2(n-s)^2 \right. \\ &\quad \left. - (4sv + n^2 - 2sn - s^2)^2 \right\} \\ &= (n^2 + 2ns - s^2)^2 - 8nv(n^2 + 2ns - s^2) + 16n^2v^2 \\ &= (n^2 + 2ns - s^2 - 4nv)^2 = \text{a square.} \end{aligned}$$

Also

$$2x^2 + 2z^2 - y^2 = \left\{ 4s^4 + 8s^3n + 8s^2n^2 - 8sn^3 + 4n^4 \right\} = \text{a square;}$$

and dividing by 4

$$s^4 + 2s^3n + 2s^2n^2 - 2sn^3 + n^4 - 2v(3s^3 + 3s^2n - sn^2 + n^3) + v^2(3s + n)^2 = \text{a square; \&}$$

$$2y^2 + 2z^2 - x^2 = \left\{ s^4 + 8s^3n + 8s^2n^2 - 8sn^3 + n^4 \right\} = \text{a square.}$$

And from these expressions we might obtain a general solution, but the calculation would be very troublesome. In order therefore to avoid a tedious calculation we shall proceed in a manner less general.

By a bare inspection of the latter of the two formulæ which remain to be made squares it will at once appear, that if we take $n = 3s$, then v will only have one dimension in that formula, and from thence we shall have

$$x = 2v(n+s) - (n^2 + s^2) = 8sv - 10s^2;$$

$$y = 2v(n-s) = 4sv \text{ and}$$

$$z = 4sv + n^2 - 2sn - s^2 = 4sv + 2s^2.$$

But as each of these expressions is, divisible by $2s$, therefore we may put

$$x = 4v - 5s, \quad y = 2v, \text{ and } z = 2v + s;$$

and by means of these

$$2x^2 + 2y^2 - z^2 = (6v - 7s)^2 = \text{a square,}$$

$$2x^2 + 2z^2 - y^2 = \left\{ \begin{aligned} &36v^2 - 72sv + 52s^2 \\ &= 4(9v^2 - 18sv + 13s^2) \end{aligned} \right\} = \text{a square,}$$

$$\text{and } 2y^2 + 2z^2 - x^2 = 48sv - 23s^2 = \text{a square.}$$

Therefore

$$9v^2 - 18sv + 13s^2 \text{ and } 48sv - 23s^2 \text{ are to be squares.}$$

Put

Put the last expression, viz. $48sv - 23s^2 = t^2$, whence
 $v = \frac{t^2 + 23s^2}{48s}$ which being written for v in the former expression, we have

$$\frac{(t^2 + 23s^2)^2}{256s^2} - 6(t^2 + 23s^2) + 13s^2 = \frac{t^4 - 50t^2s^2 + 1649s^4}{256s^2} = \text{a square};$$

therefore $t^4 - 50t^2s^2 + 1649s^4 = \text{a square}$; and this will be the case when $t = s$. But this relation of t and s would make the sum of two of the sides equal to the remaining one, and therefore no triangle would be formed. In order to obtain such a relation of t and s as will satisfy the question, put $t = r + s$; then $t^4 - 50t^2s^2 + 1649s^4 = r^4 + 4r^3s - 44r^2s^2 - 96rs^3 + 1600s^4 =$

a square. Assume $r^2 - \frac{6rs}{5} + 40s^2$ for its root, that is, put

$$r^4 + 4r^3s - 44r^2s^2 - 96rs^3 + 1600s^4 = (r^2 - \frac{6rs}{5} + 40s^2)^2 =$$

$$r^4 - \frac{12r^3s}{5} + \frac{36r^2s^2}{25} + 80r^2s^2 - 96rs^3 + 1600s^4, \text{ which}$$

after proper reduction gives $\frac{r}{s} = \frac{98}{5}$, wherefore r and s may be

expressed by any numbers in the ratio of 98 to 5. Take $r = 98$ and $s = 5$, then $t = r + s = 103$; whence

$$v = \frac{t^2 + 23s^2}{48s} = \frac{233}{5}, x = 4v - 5s = \frac{932}{5} - 25 = \frac{807}{5},$$

$$y = 2v = \frac{466}{5} \text{ and } z = 2v + s = \frac{491}{5},$$

and rejecting the common denominator 5, we may take $x = 807$, $y = 466$, and $z = 491$; and therefore the sides of a triangle that will answer are 1614, 932, and 982.

The values of x , y , and z , deduced in the preceding solution will make the expressions

$$(x+y)^2 + (x-y)^2 - z^2, (x+z)^2 + (x-z)^2 - y^2 \text{ \& } (y+z)^2 + (y-z)^2 - x^2$$

all rational squares; those expressions being respectively equal to $2x^2 + 2y^2 - z^2$, $2x^2 + 2z^2 - y^2$ and $2y^2 + 2z^2 - x^2$.

PROBLEM III.

To find two isosceles triangles such, that their perimeters and areas may be equal.

SOLUTION.

SOLUTION.

Let ACB, and DEF (fig. 151, pl. 7.) be two isosceles triangles; draw CD and FG perpendicular to AB and DE respectively.

Put $AC = BC = m^2 + n^2$, $AD = DB = m^2 - n^2$; then $2mn = CD$. Also put $DF = EF = p^2 + q^2$, $DG = GE = p^2 - q^2$; then $2pq = FG$; and because the perimeters of the triangles are equal,

$$2AC + 2AD = 2DF + 2DG, \text{ that is, } 4m^2 = 4p^2, \text{ or } m = p.$$

Again since the areas are equal

$DC \times AD = FG \times DG$, that is, $2mn(m^2 - n^2) = 2pq(p^2 - q^2)$; exterminating p by means of the equation $m = p$, we have,

$$2mn(m^2 - n^2) = 2mq(m^2 - q^2); \text{ whence } m^2 = \frac{n^3 - q^3}{n - q} = n^2 + nq + q^2;$$

$$\text{put } m = p = \frac{rn}{s} - q, \text{ that is, } m^2 = \left(\frac{rn}{s} - q\right)^2 = \frac{r^2 n^2}{s^2} - \frac{2rnq}{s} + q^2 = n^2 + nq + q^2;$$

$$\text{from whence, after proper reduction, } n = \frac{q(2rs + s^2)}{r^2 - s^2}.$$

$$\text{Take } q = r^2 - s^2, \text{ then } n = 2rs + s^2 \text{ and } m = p = \frac{rn}{s} - q = r^2 + rs + s^2;$$

$$\text{whence } AC = m^2 + n^2 = r^4 + 2r^3s + 7r^2s^2 + 2rs^3 + 2s^4$$

$$AB = 2AD = 2(m^2 - n^2) = 2r(r + 2s)(r^2 - s^2),$$

$$DF = p^2 + q^2 = 2r^4 + 2r^3s + r^2s^2 + 2rs^3 + 2s^4,$$

$$DE = 2DG = 2(p^2 - q^2) = 2rs(2r^2 + 5rs + 2s^2);$$

which are general expressions for the sides of the two isosceles triangles.

Example 1. Take $r = 2$, and $s = 1$, then $AC = 74$, $AB = 48$; $DF = 58$, and $DE = 80$; and as these numbers are all divisible by 2, we may take $AC = 37$, $AB = 24$; $DF = 29$, and $DE = 40$.

Example 2. Take $r = 3$, and $s = 1$, then $AC = 218$, $AB = 240$; $DF = 233$, and $DE = 210$.

ARTICLE XI.

Problems relating to the Twilight of Shortest Duration.

BY ASTRONOMICUS.

PROBLEM I.

TO find the day when the twilight is shortest in a given latitude.

Let P (fig. 160, pl. 9.) be the elevated pole, Z the zenith, and NPZS the meridian: let pq be part of the small circle where twilight ends (which is 18° below the horizon), and let H be the point of the horizon where the sun rises in the morning, and K the point where he passes the circle pq , bounding twilight, in the evening: then it is obvious that the difference of the horary angles SPH and SPK will be proportional to the duration of twilight.

Through the points H and K draw the great circle HOK; bisect the arc HK in m , and draw the great circle Pm; draw also the azimuth circles ZH, ZrK.

Supposing the sun's declination to remain unchanged from the time of rising in the morning to the time of passing the small circle pq in the evening, the polar distances PH and PK will be equal to one another. In the triangle ZPH, we have (No. 2, Repository, Part 3, page 11.)

Cof. ZH = cof. PZ cof. PH + fin. PZ fin. PH cof. ZPH; and in like manner, in the triangle KPZ, we have
cof. ZK = cof. PK cof. ZP + fin. PK fin. PZ cof. KPZ.

Take the difference of the equations just found, and
fin. PZ fin. PK (cof. HPZ - cof. KPZ) = cof. ZH - cof. ZK,
now cof. ZH = 0, because ZH = 90° , and - cof. ZK
= - cof. 108° = fin. 18° : also cof. HPZ - cof. KPZ =

$2 \times \text{fin.} \frac{\text{KPZ} + \text{HPZ}}{2} \times \text{fin.} \frac{\text{KPZ} - \text{HPZ}}{2}$; therefore we have

$\text{fin. PZ fin. PK fin.} \frac{\text{KPZ} + \text{HPZ}}{2} \text{fin.} \frac{\text{KPZ} - \text{HPZ}}{2} = \frac{1}{2} \text{fin. } 18^\circ.$

Because KP = PH and Km = mH, it is obvious that the great circle Pm will cut the great circle HK at right angles, and further that the angle Kpm = angle mPH. Therefore angle
Kpm

$KP_m = \frac{KPZ + KPH}{2}$. And, on account of the right angled triangle KP_m , fin. KP fin. $KP_m = \text{fin. } Km = \text{fin. } \frac{KH}{2} =$

fin. KP fin. $\frac{KPZ + HPZ}{2}$. Therefore we shall have

$$\text{fin. } PZ \text{ fin. } \frac{HK}{2} \text{ fin. } \frac{KPZ - HPZ}{2} = \frac{1}{2} \text{ fin. } 18^\circ.$$

Now $HK = HO + OK$; and it is manifest that HO is a semi-circle; therefore $HK = 180^\circ + OK$; and fin. $\frac{HK}{2} =$

fin. $(90^\circ + \frac{OK}{2}) = \text{cof. } \frac{OK}{2}$: therefore we have

$$\text{fin. } PZ \text{ cof. } \frac{OK}{2} \text{ fin. } \frac{KPZ - HPZ}{2} = \frac{1}{2} \text{ fin. } 18^\circ.$$

The difference of the horary angles KPZ and HPZ (and, consequently, the duration of twilight) will be a minimum

when fin. $\frac{KPZ - HPZ}{2}$ is a minimum: but from the last

equation fin. $\frac{KPZ - HPZ}{2}$ will be a minimum when cof. $\frac{KO}{2}$

is a maximum; for all the other quantities in that equation are constant. Therefore, it is plain, that the duration of twilight is a minimum, when the arch KO is a minimum, that is, when the great circle HOK is perpendicular to the horizon, or when it passes through Z , the pole of the horizon.

Thus the duration of twilight is a minimum in any latitude, on that day when the sun crosses the small circle bounding twilight, after sun-setting, on the same azimuth circle on which he rose in the horizon in the morning; or, which is the same thing, when the sun crosses the small circle bounding twilight, before sun-rising, on the same azimuth circle on which he will set in the horizon in the evening.

It is now easy to find the sun's declination (and, consequently, the day of the year) when the twilight is a minimum: for suppose KZH to be a great circle passing through the zenith and the points H and K , as has been shewn to be the case when the twilight is the least of all; and draw $P\mu$ perpendicular to that great circle: then because $PK = PH$, therefore $K\mu = \mu H$.

Therefore $\mu K = \frac{KZH}{2} = 99^\circ$ and $Z\mu = 9^\circ$: and in the triangle KZP , we have

$$\text{cof. } Z\mu : \text{cof. } \mu K :: \text{cof. } PZ : \text{cof. } PK :$$

$$\text{Therefore } \cos. PK = \frac{\cos. \mu K \cos. PZ}{\cos. \mu Z},$$

or, $\sin. \odot$'s declin. = $-\tan. 9^\circ \times \sin. \text{latitude}$, where the sign $(-)$ shews that the declination is south when the latitude is north, and *vice versa*.

To find the length of the shortest twilight: in the right-angled triangle $ZP\mu$, we have

$$\sin. ZP\mu = \frac{\sin. Z\mu}{\sin. PZ} = \frac{\sin. 9^\circ}{\cos. \text{latitude}}$$

and the double of the angle $ZP\mu$, being converted into time, will give the duration of the shortest twilight required.

The same result may be obtained by putting 18° (which is the least value of KO) for KO in the equation obtained above.

PROBLEM II.

To find the latitude of that place where the twilight is a minimum on a given day.

Let Z (fig. 161, pl. 9.) be zenith, NS the horizon, and H the point in the horizon where the sun sets; also let K be the point where the sun, after setting, crosses the small circle γr bounding twilight: let P be the pole, and draw the circles of declination PH and PK and the azimuth circles ZH and ZK : lastly, let a great circle pass through H and K .

Then because the day of the year is given the sun's declination will be given: therefore the polar distance $PH = PK$, will be given: therefore in the triangle HPK , of which the two sides HP and PK are constant, the less the base HK is, the less will be the horary angle HPK , which measures the duration of twilight; and when the base HK is least of all, the angle HPK will likewise be least of all. But in the triangle KZH , the side HK can never be less than the difference of the two sides ZK and ZH , that is, HK can never be less than KL , or 18° . Therefore HK will be least of all when it is equal to KL , or 18° . Therefore the duration of twilight will be least of all at that place where the pole is so situated (as at p .) that the point H may coincide with the point L , or so that pK and pL may be each equal to the distance of the sun from the elevated pole on the given day.

Thus the duration of twilight is a minimum on a given day at that place where the sun crosses the small circle 18° below the horizon, after sun-setting, on the same azimuth circle on which he set in the horizon: or, which is the same thing, where the sun crosses the small circle 18° below the horizon, before sun-rising, on the same azimuth circle on which he will rise in the horizon.

To find the latitude draw ρm perpendicular to ZK : then because $\rho K = \rho L$, therefore $Km = Lm$: therefore $Km = 9^\circ$ and $mZ = 99^\circ$. In the triangle ρZK ,

$$\text{cof. } mK : \text{cof. } mZ :: \text{cof. } \rho K : \text{cof. } \rho Z,$$

$$\text{therefore } \text{cof. } \rho Z = \frac{\text{cof. } 99^\circ}{\text{cof. } 9^\circ} \times \text{cof. } \rho K = -\tan. 9^\circ \times \text{cof. } \rho K$$

or, $\sin. \text{latitude} = -\tan. 9^\circ \times \sin. \odot$'s declination, where the sign (—) shews that the latitude is north when the sun's declination is south, and *vice versa*.

To find the length of twilight on a given day at that place where it is the least of all; we have

$$\sin. K\rho m = \frac{\sin. Km}{\sin. \rho K} = \frac{\sin. 9^\circ}{\text{cof. } \odot \text{'s declination}},$$

and the double of the angle $K\rho m$, converted into time, will give the duration of the shortest twilight on the given day.

The very least twilight over all the earth, and for all the year, is at the equator, when the sun is in the equinoctial points.

Take $\sin. q = \tan. 9^\circ \times \sin. 23^\circ 28'$; and describe two small circles on the earth, on opposite sides of the equator, and at the distance q from it: then, at every place comprehended in the zone between the two circles, the twilight will, twice in a year, be less than at any other place on the surface of the earth.

The Editor has also been favoured with papers on this subject from two other correspondents: *viz.* one from Mr. Benjamin Gompertz, containing the corrections of some inaccuracies in Emerson's Solution to prob. 54 of his Miscellanies; and the other from Mr. Surtees, containing the investigation of both cases of the general problem, and the application of the latter case to the solution of the 316th question of the Old Series of the Repository.

After what has been done above, it will be unnecessary to give either of these papers at full length; we shall therefore only insert that part which relates to the 316th question.

Mr. Surtees, by putting m and t for the sine and tangent of half the depression (9°), c for the sine of the latitude, and y for the sine of the declination; finds $c = -ty$, when the declination is constant, and the duration of twilight the shortest possible, and the sine of half the duration $= \frac{m}{\sqrt{(1-y^2)}}$; which precisely agrees with what is determined by Astronomicus in the problems above.

Now in the 316th question, the sun's declination on October 12th is $7^{\circ} 9'$ and $c = -ty = \text{fine of } 1^{\circ} 7' \frac{2}{3}$, the latitude of the place where the duration of twilight is absolutely a minimum on that day; and the fine of half the duration is $= \frac{m}{\sqrt{(1-y^2)}}$ $= \text{fine of } 9^{\circ} 4' \frac{2}{3}$, or its double $= 18^{\circ} 9' \frac{1}{3} = 1 \text{ h } 12 \frac{2}{3} \text{ m}$, the whole time of the duration of twilight in that latitude on the 12th of October.

To find the duration of twilight at the equator on any day of the year. Let P, S, (fig. 162, pl. 9.) represent the poles, PAHS the horizon, dHd a parallel of declination, and ZACN an azimuth circle passing through C, the place of the sun when 18° below the horizon. Then $\text{fin. PC} = \text{cof. declination} = \sqrt{(1-y^2)}$, and the fine of the depression, $\text{AC} = 2m \sqrt{(1-m^2)}$; and in the right angled spherical triangle PAC, $\text{fin. PC} : \text{rad.} :: \text{fin. AC} : \text{fin. } \angle \text{APC}$ $= \frac{2m}{\sqrt{(1-y^2)}} \times \sqrt{(1-m^2)}$, the fine of the whole duration at the equator, the fine of the sun's declination being y .

Now the fine of half the duration, found above, is $= \frac{m}{\sqrt{(1-y^2)}}$, or the fine of the whole duration $= \frac{2m}{\sqrt{(1-y^2)}} \times \sqrt{(1-\frac{m^2}{1-y^2})}$ and therefore the fine of the duration at the equator will be to the fine of the duration in latitude $c = -ty$, as $\frac{2m}{\sqrt{(1-y^2)}} \times \sqrt{(1-m^2)}$ to $\frac{2m}{\sqrt{(1-y^2)}} \times \sqrt{(1-\frac{m^2}{(1-y^2)})}$, or as $\sqrt{(1-m^2)}$ to $\sqrt{(1-\frac{m^2}{1-y^2})}$; where it is evident, that if y be of any value whatever, the latter expression will be less than the former, and therefore the duration of twilight in latitude $c = -ty$ is less than the time of duration at the equator. Consequently the duration of twilight is *not* a minimum at the equator at *all* times of the year, but only on the days of the equinoxes when the fine of half the duration $\frac{m}{\sqrt{(1-y^2)}}$ is $=$ to m , or the whole time $= 1$ hour and 12 minutes.

The error in finding the duration of twilight at the equator in the solution of the 316th question arises from considering the triangle PHC as a right angled great circle spherical triangle, when it is really an isosceles triangle, and the arc HC a portion of a small circle of the sphere instead of a great one.

ARTICLE XII.

Certain Fluents expressible by an Elliptic Arch.

By Mr. JAMES CUNLIFFE, R. M. College.

THE fluent of $\dot{v} \sqrt{\left(\frac{1-e^2 v^2}{1-v^2}\right)}$ is known to express the length of an elliptic arch intercepted between the conjugate axis and a parallel ordinate at the distance v from the centre: the transverse axis of the ellipse being 1, and eccentricity e .

This expression may, by various artifices, be transformed into others, the fluents whereof are, in like manner, expressible by the arch of an ellipsis.

Let us try the result of one artifice, that is, let us make the denominator $1-v^2$ a rational square. For this purpose assume $1-vx$ for the root of $1-v^2$, that is, put $1-v^2 =$

$$(1-vx)^2 = 1 - 2vx + v^2 x^2; \text{ whence } v = \frac{2x}{1+x^2},$$

$$\dot{v} = \frac{2\dot{x}(1-x^2)}{(1+x^2)^2}, \sqrt{\left(\frac{1-e^2 v^2}{1-v^2}\right)} = \sqrt{\frac{1 - \frac{4e^2 x^2}{(1+x^2)^2}}{1 - \frac{4x^2}{(1+x^2)^2}}} =$$

$$\frac{1}{1-x^2} \times \sqrt{\{1 - x^2 \times (4e^2 - 2) + x^4\}},$$

And, by means of these,

$$\dot{v} \sqrt{\left(\frac{1-e^2 v^2}{1-v^2}\right)} = \frac{2\dot{x}}{(1+x^2)^2} \times \sqrt{\{1 - x^2 \times (4e^2 - 2) + x^4\}}.$$

Therefore the fluent of any expression that can be reduced to the form $\frac{2\dot{x}}{(1+x^2)^2} \times \sqrt{\{1 - x^2 (4e^2 - 2) + x^4\}}$ may be exhibited by means of an elliptic arch.

As

As an example of the use of the foregoing conclusion, let us take the 77th question of the Repository, where it is required to

find the fluent of $\frac{x}{\sqrt{(1+x^2)^3}}$.

$$\text{In the first place } \frac{x}{\sqrt{(1+x^2)^3}} = \frac{x}{(1+x^2)^2} \sqrt{\frac{1+x^2}{1+x^2}} = \frac{x}{(1+x^2)^2} \times \sqrt{(1-x^2+x^4)}.$$

Now by making $4e^2 - 2 = 1$, or, taking $e = \frac{1}{2}\sqrt{3}$, the two expressions become identical, that is, the fluent of

$\frac{x}{\sqrt{(1+x^2)^3}}$ is expressible by half the length of the arch of an ellipse intercepted between the conjugate axis and a parallel ordinate at the distance of $\frac{2x}{1+x^2}$ from the centre: the semi-transverse axis of the ellipse being 1, and eccentricity $\frac{1}{2}\sqrt{3}$, or the transverse and conjugate axes of the ellipse being 2 and 1 respectively.

And in general the fluent of $\frac{x}{(1+x^2)^2} \sqrt{(1-ax^2+x^4)}$ is expressible by the arch of an ellipse; where a may denote any positive number whatever; and even when a denotes a negative number less than 2.

Again, let it be required to find the fluent of

$$\frac{x}{(1+x^2)^2} \sqrt{(1+x^4)}.$$

It has been shewn that the fluent of

$\frac{x}{(1+x^2)^2} \times \sqrt{\{1-x^2 \times (4e^2-2) + x^4\}}$ is expressible by half the length of the arch of an ellipse intercepted between the conjugate axis and a parallel ordinate at the distance $\frac{2x}{1+x^2}$ from the centre: the semi-transverse axis of the ellipse being 1 and eccentricity

centricity e . And it is obvious that if we put $4e^2 - 2 = 0$, or $e = \sqrt{\frac{1}{2}}$, the second term under the radical sign will disappear, and the expression then becomes $\frac{z}{(1+x^2)^{\frac{3}{2}}} \times \sqrt{(1+x^2)}$; and therefore its fluent is expressible by half the length of the arch of an ellipsis intercepted between the conjugate axis, and a parallel ordinate at the distance $\frac{2x}{1+x^2}$ from the centre; the semi-transverse axis of the ellipsis being 1 and eccentricity $\sqrt{\frac{1}{2}}$, or the transverse and conjugate axes of the ellipse are 2, and $\sqrt{2}$ respectively.

To find the fluxion of the length of the arch of a given ellipse, at any point of the curve in terms of the semi-diameter at that point.

Let a denote the semi-transverse, and b the semi-conjugate axes of the ellipse: y an ordinate thereof parallel to the conjugate axis at the distance x from the centre, and z the arch of the ellipse intercepted between the conjugate axis and ordinate y .

Then it is well known, that

$$\dot{z} = \dot{x} \sqrt{\left\{ a^2 - \left(\frac{a^2 - b^2}{a^2} \right) x^2 \right\}}.$$

Now let v denote the semi-diameter at the point where the ordinate y cuts the curve; then we shall have

$$v^2 = x^2 + y^2 = x^2 + \frac{b^2}{a^2} \times (a^2 - x^2).$$

from whence

$$x^2 \times \left(\frac{a^2 - b^2}{a^2} \right) = v^2 - b^2; \quad a^2 - \left(\frac{a^2 - b^2}{a^2} \right) \times x^2 = a^2 + b^2 - v^2;$$

$$a^2 - x^2 = a^2 - \frac{a^2 \times (v^2 - b^2)}{a^2 - b^2} = \frac{a^2}{a^2 - b^2} \times (a^2 - v^2);$$

$$\text{also } \dot{x} = \frac{a}{\sqrt{(a^2 - b^2)}} \times \frac{v\dot{v}}{\sqrt{(v^2 - b^2)}}.$$

And from hence,

$$\dot{z} = \dot{x} \sqrt{\left\{ a^2 - \left(\frac{a^2 - b^2}{a^2} \right) \times x^2 \right\}} = v\dot{v} \sqrt{\left\{ \frac{a^2 + b^2 - v^2}{(v^2 - b^2) \times (a^2 - v^2)} \right\}}$$

Q. E. I.
Put

Put $cd - cu = v^2$, then

$$v \sqrt{\left\{ \frac{a^2 + b^2 - v^2}{(v^2 - b^2)(a^2 - v^2)} \right\}} = \frac{-u\sqrt{c}}{2} \sqrt{\left\{ \frac{\frac{a^2 + b^2}{c} - d + u}{(d - \frac{b^2}{c} - u)(\frac{a^2}{c} - d + u)} \right\}}$$

Whence it appears, that fluxionary expressions of the form $u \sqrt{\left\{ \frac{m + u}{(r - u)(s + u)} \right\}}$ may be integrated by means of an elliptic arch; where m , r , and s , denote constant quantities.

Again, put $d - \frac{b^2}{c} = \frac{a^2}{c} - d$, which gives $2d = \frac{a^2 + b^2}{c}$, and

$$\frac{-u\sqrt{c}}{2} \sqrt{\left\{ \frac{\frac{a^2 + b^2}{c} - d + u}{(d - \frac{b^2}{c} - u)(\frac{a^2}{c} - d + u)} \right\}} = \frac{-u\sqrt{c}}{2} \sqrt{\left\{ \frac{d + u}{(d - \frac{b^2}{c})^2 - u^2} \right\}}$$

the fluent of which is expressible by the arch of an ellipse intercepted between the conjugate axis and a parallel ordinate at the distance

$$x = \frac{a \sqrt{(v^2 - b^2)}}{\sqrt{(a^2 - b^2)}} = \frac{a}{\sqrt{(a^2 - b^2)}} \times \sqrt{(cd - b^2 - cu)}$$

from the centre, the semi-transverse, and semi-conjugate axes of the ellipse being $a = \sqrt{(2dc - b^2)}$, and b respectively.

NEW SERIES
OF THE
MATHEMATICAL REPOSITORY.

VOL. I. PART III.

**MATHEMATICAL MEMOIRS, EXTRACTED
FROM WORKS OF EMINENCE.**

ARTICLE I.

Solutions of some Problems relative to Spherical Triangles; together with a complete Analysis of these Triangles.

BY I. L. LAGRANGE.*

IT is known that in rectilineal triangles the sides are proportional to the sines of the opposite angles, and it is easily demonstrated that this constant ratio of the sides to the sines is equal to the diameter of the circumscribing circle.

The same ratio may also be expressed by the area of the triangle, and it may be easily proved to be equal to the product of the three sides divided by twice the area.

1. But in order to express that ratio by the sides of the triangle alone, it is only necessary to consider that, by putting a, b, c for the three sides, and A, B, C for the angles opposite to them, we have by a known theorem $a^2 = b^2 + c^2 - 2bc \cos A$;

* From Journal De l'ecole Polytechnique, Vol. II. Page 270.

therefore $\text{cof. } A = \frac{b^2 + c^2 - a^2}{2bc}$, and hence

$$\sin. A = \frac{\sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}}{2bc}.$$

Therefore, if in order to simplify, we put

$$d = \sqrt{[4b^2c^2 - (b^2 + c^2 - a^2)^2]}$$

we have $\sin. A = \frac{d}{2bc}$, and hence $\frac{a}{\sin. A} = \frac{2abc}{d}$ which fraction expresses the ratio sought.

If r be put for the radius of the circle which circumscribes the triangle, and s for the surface, or area of the same triangle,

we have $\frac{a}{\sin. A} = \frac{2abc}{d} = 2r = \frac{abc}{2s}$; therefore $r = \frac{abc}{d}$, and

$$\text{and } s = \frac{abc}{4r} = \frac{d}{4}.$$

2. By developing the square of $b^2 + c^2 - a^2$, and reducing, we have

$$d = \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}$$

a formula in which the three sides a, b, c are equally concerned, as they ought to be.

But this formula may be rendered more simple and more commodious for logarithmic calculation by resolving it into factors. For

$$4b^2c^2 - (b^2 + c^2 - a^2)^2 = (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2),$$

and again resolving each of these factors into two, we have

$$d = \sqrt{[(a + b + c)(a - b + c)(a + b - c)(-a + b + c)]}.$$

These formulas are already known, and I only give them here as an introduction to investigations which are to follow.

3. Since in spherical triangles the sines of the sides are proportional to the sines of the angles opposite to them, we may be desirous to investigate the constant ratio, and to discover if it depends, as in rectilineal triangles, upon the radius of the circumscribed circle, or the area of the triangle.

Let us denote as before the three sides of a spherical triangle by a, b, c , and the three angles opposite to those sides by A, B, C .

Then by a known theorem

$$\text{cof. } a = \text{cof. } b \text{ cof. } c + \sin. b \sin. c \text{ cof. } A;$$

therefore $\text{cof. } A = \frac{\text{cof. } a - \text{cof. } b \text{ cof. } c}{\sin. b \sin. c}$; and hence

$$\sin. A = \frac{\sqrt{[\sin.^2 b \sin.^2 c - (\text{cof. } a - \text{cof. } b \text{ cof. } c)^2]}}{\sin. b \sin. c}$$

Let

Let us, to abbreviate, put

$$f = \sqrt{[\sin.^2 b \sin.^2 c - (\cos. a - \cos. b \cos. c)^2]}$$

thus we have $\sin. A = \frac{f}{\sin. b \sin. c}$, and therefore

$$\frac{\sin. a}{\sin. A} = \frac{\sin. a \sin. b \sin. c}{f}$$

an expression for the ratio sought analogous to that which has been already found for rectilineal triangles (No. 1).

As the quantity f is expressed by a square root, and may, therefore, have either the sign plus, or minus prefixed to it, we remark, that relatively to spherical triangles, it ought to be always positive; for the sides and the angles of every triangle being always less than two right angles, the sines are necessarily always positive.

4. The radical quantity f is also susceptible of reductions similar to those of No. 2. For by substituting instead of $\sin.^2 b$ and $\sin.^2 c$ their values $1 - \cos.^2 b$ and $1 - \cos.^2 c$; and afterwards reducing, we have

$$f = \sqrt{[1 - \cos.^2 a - \cos.^2 b - \cos.^2 c + 2\cos. a \cos. b \cos. c]}$$

where it appears that the three sides a, b, c are alike concerned.

In like manner the quantity under the radical sign may be resolved into factors, for we have presently

$$\begin{aligned} & \sin.^2 b \sin.^2 c - (\cos. a - \cos. b \cos. c)^2 \\ &= (\sin. b \sin. c + \cos. a - \cos. b \cos. c) \\ & \times (\sin. b \sin. c - \cos. a + \cos. b \cos. c) \\ &= [\cos. a - \cos. (b + c)] [\cos. (b - c) - \cos. a] \end{aligned}$$

but we have, in general $\cos. a - \cos. h = 2 \sin. \frac{a+h}{2} \times \sin. \frac{h-a}{2}$

therefore, resolving thus the two factors, we have

$$f = 2 \sqrt{[\sin. \frac{a+b+c}{2} \times \sin. \frac{a-b+c}{2} \times \sin. \frac{a+b-c}{2} \times \sin. \frac{-a+b+c}{2}]}$$

an expression very convenient for logarithmic calculation.

5. Let us now seek the radius of the circle circumscribing the spherical triangle. It is evident that this circle must be a small circle of the sphere, and that it will also circumscribe the rectilineal triangle formed by the chords of the arches a, b, c ;

but these chords being expressed by $2 \sin. \frac{a}{2}$, $2 \sin. \frac{b}{2}$, $2 \sin. \frac{c}{2}$ it is only necessary to substitute these quantities instead of $a, b,$

c in the expression of the radius r (No. 2), that is to say, in $\frac{abc}{d}$.

Let us put R for the radius of the circumscribed circle, and h for the value which d acquires by substituting for a, b, c as above; thus we have

$$R = \frac{8 \sin. \frac{a}{2} \times \sin. \frac{b}{2} \times \sin. \frac{c}{2}}{h}.$$

But we have $h^2 = 3^2 (\sin. \frac{a}{2} \sin. \frac{b}{2})^2 + 3^2 (\sin. \frac{a}{2} \sin. \frac{c}{2})^2 + 3^2 (\sin. \frac{b}{2} \sin. \frac{c}{2})^2 - 16 (\sin. \frac{a}{2})^4 - 16 (\sin. \frac{b}{2})^4 - 16 (\sin. \frac{c}{2})^4$; and substituting for $2(\sin. \frac{a}{2})^2, 2(\sin. \frac{b}{2})^2, 2(\sin. \frac{c}{2})^2$ their values $1 - \cos. a, 1 - \cos. b, 1 - \cos. c$ we have after proper reductions

$$\begin{aligned} h^2 &= 12 - 8 \cos. a - 8 \cos. b - 8 \cos. c \\ &\quad + 8 \cos. a \cos. b + 8 \cos. a \cos. c + 8 \cos. b \cos. c \\ &\quad - 4 \cos.^2 a - 4 \cos.^2 b - 4 \cos.^2 c \end{aligned}$$

which expression may be reduced to the following

$$4f^2 + 8(1 - \cos. a)(1 - \cos. b)(1 - \cos. c)$$

that is to say, to

$$4f^2 + 64 (\sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2})^2$$

by substituting the value of f of No. 4. Thus we have

$$R = \frac{4 \sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2}}{\sqrt{[f^2 + 16 (\sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2})^2]}}.$$

6. Now by considering the radius of the sphere which passes through the centre of the small circle circumscribing the triangle, it is evident that this radius is perpendicular to the plane of the circle, and that it meets the surface of the sphere at a point which is the pole of the same circle.

Therefore, putting ϕ for the arch which measures the distance of the circle from its pole, we have evidently $R = \sin. \phi$, therefore

sin.

$$\sin. \phi = \frac{4 \sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2}}{\sqrt{f^2 + 16 \left(\sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2} \right)^2}}.$$

from whence we derive

$$\text{cof. } \phi = \frac{f}{\sqrt{f^2 + 16 \left(\sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2} \right)^2}}$$

and hence

$$\tan. \phi = \frac{4 \sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2}}{f}.$$

Wherefore, seeing that $\sin. a = 2 \sin. \frac{a}{2} \cos. \frac{a}{2}$, and so on with respect to $\sin. b$ and $\sin. c$, we have (No. 3).

$$\frac{\sin. a}{\sin. A} = 2 \tan. \phi \cos. \frac{a}{2} \cos. \frac{b}{2} \cos. \frac{c}{2}.$$

7. If it were required to find the area of the rectilineal triangle formed by the chords of the arches a, b, c and inscribed in the circle we have just now been considering; by putting S for that area, it would only be necessary to change, in the

formula $s = \frac{abc}{4r}$ of No. 1, s into S , r into R , and a, b, c

into $2 \sin. \frac{a}{2}$, $2 \sin. \frac{b}{2}$, $2 \sin. \frac{c}{2}$ thus we should have immediately

$$S = \frac{2 \sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2}}{R}$$

or, because $R = \sin. \phi$ (No. 6)

$$S = \frac{2 \sin. \frac{a}{2} \sin. \frac{b}{2} \sin. \frac{c}{2}}{\sin. \phi}.$$

If now we consider the triangular pyramid which has this triangle for its base, and of which the vertex is at the centre of the sphere, it is evident that the height of the pyramid is

$\cos. \phi$, therefore its solidity is $\frac{S \cos. \phi}{3}$, or, substituting for S

the value we have already found,

$$2 \sin.$$

If we substitute, in place of $\tan. \frac{B+C}{2}$ its value found above (preceding No.), we have

$$\cot. \frac{\Sigma}{2} = - \frac{\tan. \frac{A}{2} \cos. \frac{b+c}{2} + \cot. \frac{A}{2} \cos. \frac{b-c}{2}}{\cos. \frac{b+c}{2} - \cos. \frac{b-c}{2}},$$

a formula, which is easily transformed to this other

$$\cot. \frac{\Sigma}{2} = \frac{\cos. \frac{b}{2} \cos. \frac{c}{2} + \sin. \frac{b}{2} \sin. \frac{c}{2} \cos. A}{\sin. \frac{b}{2} \sin. \frac{c}{2} \sin. A}.$$

If now we substitute in this formula the values $\sin. A$, and $\cos. A$ of No. 3, we shall have, by dividing the numerator and denominator by $\sin. \frac{b}{2} \sin. \frac{c}{2}$

$$\cot. \frac{\Sigma}{2} = \frac{4(\cos. \frac{b}{2} \cos. \frac{c}{2})^2 + \cos. a - \cos. b \cos. c}{f};$$

But $2(\cos. \frac{b}{2})^2 = 1 + \cos. b$, and $2(\cos. \frac{c}{2})^2 = 1 + \cos. c$,

therefore, making the necessary substitutions and inverting the fraction,

$$\tan. \frac{\Sigma}{2} = \frac{f}{1 + \cos. a + \cos. b + \cos. c}.$$

a formula of the most simple form to determine the area of a spherical triangle by means of its three sides a, b, c .

11. We have seen (No. 7) that $\frac{f}{6}$ is the solidity of the trian-

gular pyramid formed by the three radii of the sphere which terminate in the angles of a spherical triangle. Let us now consider a triangular pyramid formed by the same radii produced at pleasure in such a manner that they become p, q, r ; and that a, b, c are the arches, or angles contained by those straight lines. To have the solidity of this pyramid, it may be considered as lying on one of its faces; for instance, that which has for its sides the lines p, q , and to have a perpendicular P drawn from the other side r upon the plane of that face. It is immediately obvious, that if a is the angle contained by p and q , the area of the face which we have considered as the base of the pyramid

pyramid will be $\frac{pq \sin. a}{2}$, therefore, the solidity of the pyramid will be $\frac{Ppq \sin. a}{6}$. But if we call θ the angle which the line r makes with the plane passing through p and q , it is evident that we have $P = r \sin. \theta$, therefore, the solidity required will be $\frac{pqr \sin. a \sin. \theta}{6}$.

The angle θ is no other than the arch let fall perpendicularly from the angle A of the spherical triangle upon the opposite side a ; we may, therefore, express $\sin. \theta$ by the sines or cosines of a, b, c the sides of the triangle; but for our purpose it is sufficient to consider that this value, as well as that of $\sin. a$, being independent of the lines p, q, r , if we make $p = 1, q = 1; r = 1$ we shall have the case of the particular pyramid we propose to consider; and of which the solidity is $\frac{f}{6}$. Hence it fol-

lows, that we have $\sin. a \sin. \theta = f$, and therefore $\frac{pqr f}{6}$ for the

solidity of the triangular pyramid in which the three sides, or edges that form any one of the solid angles are p, q, r , and the angles contained by those three sides are A, B, C .

This expression, for the solidity of any triangular pyramid, by means of its three sides and contained angles, is evidently very simple and commodious for calculation, more especially if we employ the value of f , as expressed by factors; in No. 4; and it may be very useful to determine the solid contents of all bodies bounded by planes, seeing that they may always be resolved into triangular pyramids, as a polygon is resolved into triangles.

12. We may add, that since we have found $\sin. a \sin. \theta = f$, we shall have $\sin. \theta = \frac{f}{\sin. a}$. We may thus determine by this formula the perpendicular θ in a spherical triangle, of which a is the base, and b, c the two sides.

13. I have resolved the preceding problems, only to shew the origin and use of some remarkable formulas, and more especially of that function which I have denoted by f , and which deserves in a particular manner the attention of analysts, on account of its different applications. I now proceed to general considerations relating to spherical trigonometry, viewed analytically.

Analytical resolutions of spherical triangles were at first only
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simple applications of algebra to geometrical constructions. It was afterwards thought sufficient to establish some fundamental propositions by the aid of geometry, and all the formulas of spherical trigonometry have been deduced from equations derived from those propositions. The most elegant production of this kind is a Memoir of Euler's, entitled *Trigonometria spherica uniuersa ex primis principiis derivata*, and printed in the Petersburg Acts for the year 1779, in which there is found a complete system of trigonometrical formula's, founded solely upon three equations. But may not the system be yet more simplified by reducing it to one fundamental equation?

Such a reduction will complete the analytical theory of spherical triangles; for, in Analysis, perfection consists in employing the smallest possible number of principles, and in deducing from these all the truths they contain by the powers of that method alone: on the contrary, in the synthetic method of lines, it consists in demonstrating separately every proposition in the most simple manner, by means of other propositions previously demonstrated.

The late de Gua had conceived the idea of making all spherical trigonometry to depend upon a single general property of spherical triangles; but the Memoir he has given upon the subject, in the volume of the Academy of Sciences for 1783, contains calculations so complicated, that they seem better adapted to shew the inconvenience of his method than to cause it to be adopted.

I here propose to myself the same object, and I proceed to give a concise view of all the formulas of spherical trigonometry, deducing them by simple transformations from a single equation, given by the nature of spherical triangles.

Let us assume, as De Gua has done, the equation (No. 3.)

$$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A$$

in which a, b, c are the three sides or arches of the triangle, and A the angle opposite to the side a .

This equation is easily demonstrated by the simple consideration of two rectilinear triangles formed, the one by the two tangents to the arches b, c , and by the straight line which joins the extremities of these tangents, and the other by the same line and the two secants of the same arches; for it is obvious, that the two tangents form between them the angle A which is contained by the arches b and c , and that the two secants form between them the angle a which is the side of the spherical triangle opposite to the angle A . Thus, calling h the side common to these two triangles, we have immediately, by a known theorem concerning rectilinear triangles, this equation,

$$h^2 = \tan.^2 b + \tan.^2 c - 2 \tan. b \tan. c \cos. A$$

for the former triangle, and this other equation,

$$h^2 = \sec.^2 b + \sec.^2 c - 2 \sec. b \sec. c \cos. a$$

for the latter triangle.

Hence

Hence we find

$$\begin{aligned} & \tan.^2 b + \tan.^2 c - 2 \tan. b \tan. c \cos. A \\ & = \sec.^2 b + \sec.^2 c - 2 \sec. b \sec. c \cos. a. \end{aligned}$$

But we have evidently

$$\sec.^2 b - \tan.^2 b = 1, \text{ also } \sec.^2 c - \tan.^2 c = 1,$$

therefore, the equation becomes

$$\sec. b \sec. c \cos. a = 1 + \tan. b \tan. c \cos. A,$$

hence, substituting for $\sec. b$, $\sec. c$, $\tan. b$, $\tan. c$, their values

$\frac{1}{\cos. b}$, $\frac{1}{\cos. c}$, $\frac{\sin. b}{\cos. b}$, $\frac{\sin. c}{\cos. c}$, we have the fundamental equation,

$$\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A \dots \dots \dots (A)$$

As, by hypothesis, the only condition supposed to have place among the four quantities a , b , c , A is that a , b , c are the three sides of a triangle, and A the angle opposite to the side a ; it follows, that by putting B and C for the angles opposite to the sides b and c , we shall have similar equations relatively to those angles, by simply changing A into B , or into C , provided that at the same time we change a into b , or into c .

15. Now if we deduce the value of $\cos. A$ from the preceding equations, and thence that of $\sin. A$ we shall have, as has been already found in No. 3,

$$\frac{\sin. a}{\sin. A} = \frac{\sin. a \sin. b \sin. c}{f}$$

where the quantity f is a function of a , b , c into which these three quantities enter similarly, so that it remains the same, whatever permutations be made among them. Therefore, in changing a into b , and b into a , the second member of the equation will be the same as before; hence we have this other equation,

$$\frac{\sin. a}{\sin. A} = \frac{\sin. b}{\sin. B} \dots \dots \dots (B)$$

This is what is called the common analogy of the sines, and it is obvious, that by changing a and c into A and C , we have in

like manner $\frac{\sin. a}{\sin. A} = \frac{\sin. c}{\sin. C}$.

16. Let us resume the equation (A) of No. 14, $\cos. a = \cos. b \cos. c + \sin. b \sin. c \cos. A$; by changing a into c , and A into C , we have in like manner $\cos. c = \cos. a \cos. b + \sin. a \sin. b \cos. C$; substituting this value of $\cos. c$ in the first equation it becomes $\cos. a = \cos. a \cos.^2 b + \sin. a \sin. b \cos. b \cos. C$

(b 2)

+

$+\sin. b \sin. c \cos. A$; that is to say, $\cos. a \sin.^2 b = \sin. a \sin. b \cos. b \cos. C + \sin. b \sin. c \cos. A$, and dividing by $\sin. b$

$$\cos. a \sin. b = \sin. a \cos. b \cos. C + \sin. c \cos. A.$$

Let us substitute for $\sin. c$ its value $\frac{\sin. a \sin. C}{\sin. A}$ found from the equation (B) of the preceding No. by changing b into c , and B into C ; let us then divide by $\sin. a$, and put $\cot. a$ and $\cot. A$ instead of $\frac{\cos. a}{\sin. a}$ and $\frac{\cos. A}{\sin. A}$; thus, we shall have the equation,

$$\cot. a \sin. b = \cot. A \sin. C + \cos. b \cos. C \dots (C).$$

17. Now the equation found above, viz. $\cos. a \sin. b = \sin. a \cos. b \cos. C + \sin. c \cos. A$, by changing a into b and A into B becomes $\sin. a \cos. b = \cos. a \sin. b \cos. C + \sin. c \cos. A$. Substituting this value of $\sin. a \cos. b$ in the same equation, we have $\cos. a \sin. b = \cos. a \sin. b \cos.^2 C + \sin. c \cos. B \cos. C + \sin. c \cos. A$; that is to say, $\cos. a \sin. b \cos.^2 C = \sin. c (\cos. B \cos. C + \cos. A)$: let us now substitute for $\sin. c$ its value $\frac{\sin. b \sin. C}{\sin. B}$ derived from equation (B) by changing a

into c and A into C , then divide by $\sin. b \sin. C$ and multiply by $\sin. B$; thus, we have $\cos. a \sin. B \sin. C = \cos. B \cos. C + \cos. A$, that is

$$\cos. A = -\cos. B \cos. C + \sin. B \sin. C \cos. a \dots (D).$$

18. This formula, it appears, is in every respect similar to the formula (A) of No. 14. from which we set out; the angles A, B, C have now taken the place of the sides a, b, c , and the contrary; also the cosines have become negative, the sines remaining positive, from which we may infer that the sides have become the supplements of the angles, and the angles the supplements of the sides. Thus all the formulas which result from the formula (A) are also true when the same changes are made in them. Hence we derive this known property of spherical triangles, namely, that any spherical triangle may be changed into another, the sides and the angles of which are respectively the supplements of the angles and the sides of the former; and it is known that the new triangle, (called the supplemental triangle,) is that which is formed upon the sphere by the three poles of arches which form the sides of the given triangle in joining these poles by arches of great circles, which may easily be demonstrated by a very simple construction.

19. The four equations (A), (B), (C), (D), which we have found, contain the solution of all questions in spherical trigonometry;

metry; for as a spherical triangle consists of but six parts, the three sides, and the three angles, and as three of these are sufficient to determine the triangle, it is evident that the most simple relations can only take place among four of those parts: But all the different combinations that can be made of six things, taken four by four, are the following:

1. Between the three sides and an angle: this relation is given by the equation (A).

2. Between two sides and two angles, which may be respectively opposite to the two sides, or the one opposite, and the other adjacent to the same side, making two cases. The relation between two sides, and the two opposite angles, is given by equation (B).

3. Between two sides and two angles, of which one is opposite, and the other adjacent to the same side: this relation is contained in equation (C).

4. Between the three angles and a side: this relation is given by equation (D).

20. If A be supposed a right angle, the preceding equations become more simple, and give these other equations;

$$\text{cof. } a = \text{cof. } b \text{ cof. } c$$

$$\sin. a = \frac{\sin. b}{\sin. B}$$

$$\cot. a = \cot. b \text{ cof. } C$$

$$\text{cof. } a = \cot. B \cot. C,$$

and if C be supposed a right angle, the equations (C) and (D) give these other two:

$$\cot. A = \cot. a \sin. b,$$

$$\cot. A = \sin. B \text{ cof. } a.$$

These six equations give immediately the solution of all cases of right-angled spherical triangles; and as their form readily admits of the application of logarithms, they are commonly employed in Trigonometry, by resolving all triangles into right-angled triangles, by letting fall a perpendicular. But all the cases may be also resolved by the four general equations, which may be reduced into factors by means of transformations, which we now proceed to explain.

21. The equation (A) between the three sides a, b, c , and an angle A opposite to the side a may serve to determine: 1st, A by a, b, c ; 2d, a by b, c , and A; 3d, b by a, c , and A.

1st, To determine A by a, b, c , we have

$$\text{Cof. } A = \frac{\text{cof. } a - \text{cof. } b \text{ cof. } c}{\sin. b \sin. c}$$

Hence

Hence we find

$$\begin{aligned}
 1 + \operatorname{cof.} A &= 2 \left(\operatorname{cof.} \frac{A}{2} \right)^2 = \frac{\operatorname{cof.} a - \operatorname{cof.} (b + c)}{\sin. b \sin. c} \\
 &= \frac{2 \sin. \frac{b + c - a}{2} \sin. \frac{b + c + a}{2}}{\sin. b \sin. c} \\
 1 - \operatorname{cof.} A &= 2 \left(\sin. \frac{A}{2} \right)^2 = \frac{\operatorname{cof.} (b - c) - \operatorname{cof.} a}{\sin. b \sin. c} \\
 &= \frac{2 \sin. \frac{a - b + c}{2} \sin. \frac{a + b - c}{2}}{\sin. b \sin. c}
 \end{aligned}$$

Therefore,

$$\tan. \frac{A}{2} = \sqrt{\frac{\sin. \frac{a + b - c}{2} \sin. \frac{a - b + c}{2}}{\sin. \frac{b + c + a}{2} \sin. \frac{b + c - a}{2}}} \dots (a)$$

2d, To determine a by b, c , and A , we have

$$\operatorname{cof.} a = \operatorname{cof.} b \operatorname{cof.} c + \sin. b \sin. c \operatorname{cof.} A.$$

It does not appear that this equation can be resolved directly into factors, so as to admit of the application of logarithms; it may, however, be done by means of a subsidiary angle. For if we put

$$\tan. c \operatorname{cof.} A = \tan. \phi \dots \dots \dots (b)$$

we have $\sin. c \operatorname{cof.} A = \operatorname{cof.} c \tan. \phi$, therefore, $\operatorname{cof.} a = \operatorname{cof.} c (\operatorname{cof.} b + \sin. b \tan. \phi)$; but $\operatorname{cof.} b + \sin. b \tan. \phi = \frac{\operatorname{cof.} b \operatorname{cof.} \phi + \sin. b \sin. \phi}{\operatorname{cof.} \phi} = \frac{\operatorname{cof.} (b - \phi)}{\operatorname{cof.} \phi}$, therefore,

$$\operatorname{cof.} a = \frac{\operatorname{cof.} c \operatorname{cof.} (b - \phi)}{\operatorname{cof.} \phi} \dots \dots \dots (c)$$

It is not difficult to perceive that this transformation comes to the same thing as the resolving of the triangle into right-angled triangles, by a perpendicular let fall from the angle B upon the side b ; and that ϕ is the segment of the side adjacent to the angle A .

3d, To determine b by a, c , and A , it is necessary to substitute in the principal equation (A), $\sqrt{1 - \sin.^2 b}$ for $\operatorname{cof.} b$, then to raise it to the second power, in order to make the radical quantity disappear, and lastly, to find the value of $\sin. b$ by the resolution of an equation of the second degree; by this process we have $\sin. b$ expressed by a complicated formula which cannot be applied to logarithmic calculation. But the transformation already employed

employed is also applicable to this case : for having determined the angle ϕ by equation (b) the equation (c) becomes

$$\text{cof. } (b - \phi) = \frac{\text{cof. } a \text{ cof. } \phi}{\text{cof. } c} \dots\dots\dots (d).$$

22. The equation (B) between two sides a and b , and the opposite angles A and B may serve 1st to determine A by a, b, B ; and 2d to determine a by b, A, B .

1st, To determine A by a, b, B we have

$$\sin. A = \frac{\sin. a \sin. B}{\sin. b}$$

2d, To determine a by b, A, B we have

$$\sin. a = \frac{\sin. b \sin. A}{\sin. B} \dots\dots\dots (f)$$

These formulas admit of the application of logarithms without further transformation.

23. The equation (C) between two sides a and b , and two angles A, C the first opposite, the second adjacent to the side a , may serve 1st to determine A by a, b, C ; 2d, to determine a by b, A, C ; 3d to determine C by a, b, A ; 4th to determine b by a, A, C .

1st, To determine A by a, b, C we have this equation,

$$\cot. A = \frac{\cot. a \sin. b}{\sin. C} - \cot. C \text{ cof. } b;$$

and to reduce it into factors, we assume $\frac{\cot. a}{\cot. C} = \cot. \phi$

or,

$$\tan. a \text{ cof. } C = \tan. \phi \dots\dots\dots (g)$$

therefore, $\cot. a = \cot. C \cot. \phi$; and substituting this value we have $\cot. A = \cot. C (\cot. \phi \sin. b - \cot. b)$

$$= \frac{\cot. C}{\sin. \phi} (\cot. \phi \sin. b - \sin. \phi \cot. b) = \frac{\cot. C \sin. (b - \phi)}{\sin. \phi}; \text{ theref,}$$

$$\cot. A = \frac{\cot. C \sin. (b - \phi)}{\sin. \phi} \dots\dots\dots (h)$$

This reduction comes also to the same thing, as the resolving of the triangle into two right-angled triangles, by a perpendicular from the angle B upon the opposite side b , and the subsidiary angle ϕ is the segment of the side adjacent to the angle C .

2d, To determine a by b, A, C we have this equation,

$$\cot. a = \frac{\cot. A \sin. C}{\sin. b} + \cot. b \text{ cof. } C$$

and

and to reduce it into factors, we make

$$\frac{\cot. A}{\cos. b} = \tan. \varphi \dots\dots\dots (i)$$

therefore, substituting in the equation for $\cot. A$ its value

$$\tan. \varphi \cos. b \text{ we have } \cot. a = \cot. b (\sin. C \tan. \varphi + \cos. C) = \frac{\cot. b}{\cos. \varphi} (\sin. C \sin. \varphi + \cos. C \cos. \varphi) = \frac{\cot. b \cos. (C - \varphi)}{\cos. \varphi}, \text{ theref.}$$

$$\cot. a = \frac{\cot. b \cos. (C - \varphi)}{\cos. \varphi} \dots\dots\dots (k)$$

This reduction is yet the same as the resolving of the triangle into two by a perpendicular from the angle C upon the side c , and the subsidiary angle φ is the segment of the angle C adjacent to the side b .

2d To determine C by a, b, A , it is necessary to deduce from equation (C) the value of $\sin. C$ or of $\cos. C$ by the resolution of an equation of the second degree; thus, we obtain an expression which contains a radical quantity. But the preceding transformation is equally applicable to this case; for having found the angle φ by the equation (i), we find, from equation (k), that

$$\cos. (C - \varphi) = \frac{\cot. a \cos. \varphi}{\cot. b} \dots\dots\dots (l)$$

4th, And to determine b by a, A, C , it is necessary to find, from the same equation (C) the value of $\sin. b$, or $\cos. b$ by resolving an equation of the second degree. But the transformation employed in the first case applies also here; for having found the subsidiary angle φ from equation (g), we get from equation (h)

$$\sin. (b - \varphi) = \frac{\cot. A \sin. \varphi}{\cot. C} \dots\dots\dots (m)$$

24. Finally, The equation (D) between the three angles A, B, C and the side a serves to determine, 1st, a by A, B, C ; 2d, A by a, B, C ; 3d, B by a, A, C . As these three cases correspond to those which depend on equation (A), to which equation (D) is by transformation rendered in all respects similar, we may, therefore, apply to them directly the formula of No. 21, which we have given for the others, by substituting the supplements of the angles A, B, C instead of the sides a, b, c and the supplements of the sides a, b, c instead of the opposite angles A, B, C . (No. 18.)

Thus, 1st, to determine a by A, B, C , the equation (a) of (No. 21,) gives the transformed equation:

cot.

$$\cot. \frac{a}{2} = \sqrt{\left(\frac{\cot. \frac{A+B-C}{2} \cot. \frac{A-B+C}{2}}{-\cot. \frac{B+C+A}{2} \cot. \frac{B+C-A}{2}} \right)} \dots\dots\dots (n)$$

2d. To determine A by a , B, C, we make the same substitutions in the formulas (b) and (c) of the same No. and taking instead of the angle ϕ , its complement, we have these two equations;

$$\tan. C \cot. a = \cot. \phi \dots\dots\dots (o)$$

$$\cot. A = \frac{\cot. C \sin. (B - \phi)}{\sin. \phi} \dots\dots\dots (p)$$

3d. The same equations serve to determine B by a , A, C; for having found ϕ by equation (o), equation (p) gives

$$\sin. (B - \phi) = \frac{\cot. A \sin. \phi}{\cot. C} \dots\dots\dots (q)$$

25. By these formulas we may find directly any one of the sides or angles by three given parts, which may be either sides or angles; now this constitutes the whole theory of spherical triangles. But when it is required to find at the same time two sides by means of the third side and the two opposite angles, or two angles by the third angle and the two opposite sides, it is more convenient to employ Neper's Rules. The following seems to be the most simple method of investigating them.

We have found in No. 21,

$$\tan. \frac{A}{2} = \sqrt{\left(\frac{\sin. \frac{a+b-c}{2} \sin. \frac{a-b+c}{2}}{\sin. \frac{b+c+a}{2} \sin. \frac{b+c-a}{2}} \right)}$$

In like manner, by changing A into B, and a into b , we have for the angle B

$$\tan. \frac{B}{2} = \sqrt{\left(\frac{\sin. \frac{a+b-c}{2} \sin. \frac{b-a+c}{2}}{\sin. \frac{b+c+a}{2} \sin. \frac{a-b+c}{2}} \right)}$$

therefore, taking the product of these two equations,

$$\tan. \frac{A}{2} \tan. \frac{B}{2} = \frac{\sin. \frac{a+b-c}{2}}{\sin. \frac{a+b+c}{2}}$$

In this equation let B be changed into C and b into c , and we have

$$\tan. \frac{A}{2} \tan. \frac{C}{2} = \frac{\text{fin. } \frac{a-b+c}{2}}{\text{fin. } \frac{a+b+c}{2}},$$

and by changing in the same equation A into C and a into c ,

$$\tan. \frac{B}{2} \tan. \frac{C}{2} = \frac{\text{fin. } \frac{b+c-a}{2}}{\text{fin. } \frac{a+b+c}{2}}.$$

By adding the two last equations together we have

$$\left(\tan. \frac{A}{2} + \tan. \frac{B}{2} \right) \tan. \frac{C}{2} \left\{ \begin{aligned} &= \frac{\text{fin. } \frac{a-b+c}{2} + \text{fin. } \frac{b+c-a}{2}}{\text{fin. } \frac{a+b+c}{2}}, \\ &= \frac{\text{fin. } \frac{c}{2} \text{ cof. } \frac{a-b}{2}}{\text{fin. } \frac{a+b+c}{2}}, \end{aligned} \right.$$

and by subtracting the one from the other

$$\left(\tan. \frac{A}{2} - \tan. \frac{B}{2} \right) \tan. \frac{C}{2} \left\{ \begin{aligned} &= \frac{\text{fin. } \frac{a-b+c}{2} - \text{fin. } \frac{b+c-a}{2}}{\text{fin. } \frac{a+b+c}{2}}, \\ &= \frac{\text{cof. } \frac{c}{2} \text{ fin. } \frac{a-b}{2}}{\text{fin. } \frac{a+b+c}{2}}. \end{aligned} \right.$$

On the other hand we have

$$1 + \tan. \frac{A}{2} \tan. \frac{B}{2} \left\{ \begin{aligned} &= \frac{\text{fin. } \frac{a+b+c}{2} + \text{fin. } \frac{a+b-c}{2}}{\text{fin. } \frac{a+b+c}{2}}, \\ &= \frac{\text{fin. } \frac{a+b}{2} \text{ cof. } \frac{c}{2}}{\text{fin. } \frac{a+b+c}{2}} \end{aligned} \right.$$

$$1 - \tan. \frac{A}{2} \tan. \frac{B}{2} \left\{ \begin{aligned} &= \frac{\text{fin. } \frac{a+b+c}{2} - \text{fin. } \frac{a+b-c}{2}}{\text{fin. } \frac{a+b+c}{2}} \\ &= \frac{\text{fin. } \frac{c}{2} \text{ cof. } \frac{a+b}{2}}{\text{fin. } \frac{a+b+c}{2}} \end{aligned} \right.$$

Therefore, because $\tan. \frac{A+B}{2} = \frac{\tan. \frac{A}{2} + \tan. \frac{B}{2}}{1 + \tan. \frac{A}{2} \tan. \frac{B}{2}}$, we have

these two equations

$$\tan. \frac{A+B}{2} \tan. \frac{C}{2} = \frac{\text{cof. } \frac{a-b}{2}}{\text{cof. } \frac{a+b}{2}} \dots\dots\dots (r)$$

$$\tan. \frac{A-B}{2} \tan. \frac{C}{2} = \frac{\text{fin. } \frac{a-b}{2}}{\text{fin. } \frac{a+b}{2}} \dots\dots\dots (s)$$

Similar formulas may be deduced from equation (n) of No. 24; and without making a new calculation it is only necessary to change the sides a, b, c , into the supplements of the angles A, B, C ; and these angles into the supplements of the sides. Thus we have immediately these other two equations:

$$\tan. \frac{a+b}{2} \cot. \frac{c}{2} = \frac{\text{cof. } \frac{A-B}{2}}{\text{cof. } \frac{A+B}{2}} \dots\dots\dots (t)$$

$$\tan. \frac{a-b}{2} \cot. \frac{c}{2} = \frac{\text{fin. } \frac{A-B}{2}}{\text{fin. } \frac{A+B}{2}} \dots\dots\dots (u)$$

By these four equations we can determine directly, and without the aid of a subsidiary angle, the two angles A and B by the opposite sides a and b and the included angle C , or the two sides by the opposite angles and the third side.

26. Before concluding this memoir, it may be proper to treat shortly of the comparison of spherical triangles with rectilinear triangles. By taking, as is usual, the radius of the sphere,

for unity, it is evident that the arches which form the spherical triangle express immediately angles of which the sines and cosines are found in the tables. If the radius of the sphere is not unity then, to have the angular value of the sides of the triangle, their absolute values must be divided by the radius. Thus, if α, β, γ , are the absolute lengths of the arches which form a spherical triangle upon the surface of a sphere whose radius is r , we have $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$, for the angles corresponding to those arches,

and these quantities are now to be taken for the sides denoted formerly by a, b, c , the letters A, B, C , denote the angles opposite to the arches α, β, γ , in the triangle under consideration.

Now if the radius of the sphere become infinitely great, its surface becomes a plane, and the spherical triangle becomes rectilinear; hence it follows that if in the formulas for spherical tri-

angles we substitute every where $\frac{\alpha}{r}, \frac{\beta}{r}, \frac{\gamma}{r}$, in place of a, b, c , and

afterwards suppose r infinitely great, and, having reduced into series the sines, and cosines of these angles, we reject the terms

which vanish by considering that $\frac{1}{r} = 0$, we shall have formulas

relating to rectilinear triangles in which α, β, γ , are the sides, and A, B, C , the opposite angles.

Thus if $V = 0$ is an equation expressing the relation that subsists among the sines and cosines of a, b, c , and of A, B, C , we

must substitute for $\sin. a, \frac{\alpha}{r} - \frac{\alpha^3}{2 \cdot 3 r^3} + \&c.$; and for $\cos. a$,

$1 - \frac{\alpha^2}{2 r^2} + \frac{\alpha^4}{2 \cdot 3 \cdot 4 r^4} - \&c.$ and for $\sin. b, \cos. b, \sin. c, \cos. c$,

like values, by changing α into β, γ ; and reducing the whole into a series according to the descending powers of r : we thus obtain

$$V = \frac{P}{r^m} + \frac{Q}{r^{m+1}} + \frac{R}{r^{m+2}} + \&c.$$

which, when the triangle is rectilinear, gives $P = 0$; for the equation $V = 0$, becomes by multiplying by r^m ,

$$P + \frac{Q}{r} + \frac{R}{r^2} + \&c. = 0.$$

and making $\frac{1}{r} = 0$ we have $P = 0$.

In this way may the rules of rectilinear trigonometry be derived from the fundamental equations of spherical trigonometry.

This

This, however, would be to demonstrate that which is simple by that which is compound, and, therefore, of no use except as an exercise in calculation; we shall only remark that equation (D) of No. 17, gives directly this other equation, $\text{cof. } A = \sin. B \sin. C - \text{cof. } B \text{ cof. } C$; or $\text{cof. } A = - \text{cof. } (B - C)$, and, therefore, $A = 2D - B - C$, that is, $A + B + C = 2D$, where D denotes a right angle, which is a known property of rectilinear triangles.

27. Now if the radius of the sphere, instead of being infinitely great, is only very great, the spherical triangle will only be nearly rectilinear; and here, as the angles a, b, c , which correspond to the sides become very small, the common trigonometrical tables are not sufficiently precise for the calculation of the sides and angles. In this case, therefore, spherical triangles may be treated with advantage, as rectilinear triangles, by taking into account the small correction that results from their difference.

This case always occurs in the calculation of triangles formed upon the earth's surface for the measurement of an arch of the meridian; on these triangles the quantities β, γ , are the length of the arches, and r is the earth's radius.

To determine the correction we resume equation ($\bar{A}$) of No. 14, which serves as the basis of spherical trigonometry, and which gives

$$\text{Cof. } A = \frac{\text{cof. } a - \text{cof. } b \text{ cof. } c}{\sin. b \sin. c}.$$

Making in the second member of this equation the substitutions indicated in last number, and stopping at the terms divided by r^4 , we have

$$\text{Cof. } A = \frac{\frac{\beta^2 + \gamma^2 - a^2}{2r^2} + \frac{a^4 - \beta^4 - \gamma^4}{2 \cdot 3 \cdot 4 r^4} - \frac{\beta^2 \gamma^2}{4r^2}}{\frac{\beta \gamma}{r^2} \left(1 - \frac{\beta^2 + \gamma^2}{2 \cdot 3 r^2} \right)},$$

multiplying the numerator and denominator of the fraction by r^2 , and substituting the factor $1 + \frac{\beta^2 + \gamma^2}{2 \cdot 3 r^2}$ instead of the divi-

for $1 - \frac{\beta^2 + \gamma^2}{2 \cdot 3 r^2}$, we have, neglecting such terms as are divided

by powers above r^2

$$\begin{aligned} \text{Cof. } A = & \frac{\beta^2 + \gamma^2 - a^2}{2\beta\gamma} + \frac{a^4 - \beta^4 - \gamma^4}{2 \cdot 3 \cdot 4 \beta \gamma r^2} - \frac{\beta^2 \gamma^2}{4 \beta \gamma r^2} \\ & + \frac{(\beta^2 + \gamma^2 - a^2)(\beta^2 + \gamma^2)}{3 \cdot 4 \beta \gamma r^2}, \end{aligned}$$

=

$$= \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\alpha^2\gamma^2 - 2\beta^2\gamma^2}{2 \cdot 3 \cdot 4\beta\gamma r^2}.$$

By making $\frac{1}{r} = 0$, the angle A becomes the angle opposite to the side α in the rectilinear triangle of which α, β, γ , are the sides. Let us denote that angle by A' , we have thus

$$\text{Cof. } A' = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma}$$

$$\text{and hence } \sin.^2 A' = \frac{2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4}{4\beta^2\gamma^2}$$

as we have seen already (No. 1 and 2); therefore, substituting these values in the preceding equation it becomes

$$\text{Cof. } A = \text{cof. } A' - \frac{\beta\gamma \sin.^2 A'}{2 \cdot 3r^2};$$

but, in the rectilinear triangle of which α, β, γ , are the sides, it is manifest that $\frac{\beta\gamma \sin. A'}{2}$ expresses its area, therefore, if θ be put to denote that area, we have

$$\text{Cof. } A = \text{cof. } A' - \frac{\theta \sin. A'}{3r^2},$$

hence we have, to quantities of the order, $\frac{1}{r^4}$ nearly,

$$A = A' + \frac{\theta}{3r^2}.$$

And as by changing the side α into β or γ , the angle A changes to B or C, if we design by B' and C' the angles opposite to the sides β and γ in the rectilinear triangle, we have in like manner

$$B = B' + \frac{\theta}{3r^2},$$

$$C = C' + \frac{\theta}{3r^2},$$

for the quantity θ , or the area of the rectilinear triangle is a function in which the sides are alike concerned, and which, therefore, is the same, whatever interchanges are made among the quantities that compose it.

Therefore, when a spherical triangle is traced upon the surface of a sphere, the radius of which is very great, if a rectilinear triangle is formed having its sides of the same length as those of the spherical

spherical triangle, the angles of the latter shall be equal to the corresponding angles of the rectilinear triangle, when each is augmented by the quantity $\frac{\theta}{3r^2}$, (θ being the area of the rectilinear triangle) reduced to the measure of an angle, which is done by taking for unity the angle corresponding to an arch equal to the radius.

28. If we add together the three equations $A = A' + \frac{\theta}{3r^2}$, $B = B' + \frac{\theta}{3r^2}$, $C = C' + \frac{\theta}{3r^2}$, we have $A + B + C = A' + B' + C' + \frac{\theta}{r^2}$, but we know that $A' + B' + C' = 2D$ where D denotes a right angle; therefore,

$$\frac{\theta}{r^2} = A + B + C - 2D,$$

Hence we may conclude, that by subtracting from each angle of a spherical triangle one-third of the excess of the sum of the three angles above two right angles, we shall have the three angles of a rectilinear triangle, the sides of which are equal in length to those of the spherical triangle. Thus we may treat the latter as a rectilinear triangle, and obtain results which will be accurate

to quantities nearly of the order $\frac{1}{r^4}$.

This beautiful theorem is due to Legendre, who gave it at first without demonstration in the Memoirs of the Academy of Sciences for 1787; he has since demonstrated it, in a manner differing but little from the preceding, in a Memoir upon the method of determining the length of a quadrant of the meridian. As it may be of great use in the calculation of spherical triangles which differ but little from rectilinear triangles, we have judged that the reader would be pleased to find it here.

ARTICLE II.

An Essay of Numerical Analysis, on the Transformation of Fractions.*

BY I. L. LAGRANGE.

1. **L**ET us consider the fraction $\frac{B}{A}$ which is supposed less than unity, and reduced to the lowest terms, so that the numbers A and B are prime to each other. If it were required to transform the fraction into another of which the numerator or the denominator were given, it is evident that this would be strictly possible, only when the new numerator or denominator was a multiple of the numerator or denominator. If, however, no more than an approximation be required, the problem may always be resolved, and the new fraction must be found as nearly equal to the given fraction as is possible.

2. Thus if the new fraction be designed by $\frac{m}{a}$, in which m or a is supposed given, the problem consists in finding a or m , so that the difference between $\frac{B}{A}$ and $\frac{m}{a}$ shall be the least possible.

But this difference is $\frac{Ba - Am}{Aa}$, therefore, a , or m must be found such, that the number $Ba - Am$ becomes the smallest possible; this will be effected by taking for a the quotient of Am divided by B , for m the quotient of Ba divided by A , for then the value of $Ba - Am$ will be the remainder of either division, and therefore less than the divisor.

3. But it must be here observed, that the remainder of a division may be either positive or negative, according as we take for the quotient, a number which being multiplied by the divisor produces the nearest number less, or the nearest number greater than the dividend. In common arithmetic division is so performed, that the remainders are positive, but in the general theory of numbers we may employ either positive or negative remainders, and thus the remainder may be always less than half the divisor, for when the positive remainder is greater than that half, if we add unity to the quotient, we must at the same time diminish the re-

* From Journal De l'ecole Polytechnique, Vol. II.

by the divisor, thus we shall have a different remainder, which will be negative, and less than half the divisor.

For the sake of simplicity we may call (*division en defect*) (*division en dedans*) that in which the remainder is positive, and *division en excess*, (*division en dehors*,) that which gives a negative remainder, because, in the first case, the product of the quotient by the divisor is less than the dividend, and, in the other, the product exceeds it.

4. Therefore, let $Ba - Am = \pm C$, so that $\pm C$ shall be the remainder of the division of Ba by A , and m the quotient, or $\pm C$ the remainder of the division of Am by B , and a the quotient; hence $\frac{B}{A} - \frac{m}{a} = \pm \frac{C}{Aa}$, and consequently $\frac{B}{A} = \frac{m}{a} \pm \frac{C}{Aa}$.

We may therefore treat in the same manner the fraction $\frac{C}{A}$, in which C is always necessarily less than A , and reduce it to another known fraction $\frac{n}{b}$, of which the numerator or the denominator is given, and which approaches the nearest possible to the same fraction. We thus make $Cb - An = \pm D$ where $\pm D$ is the remainder of the division of Cb by A , and n the quotient, if the numerator b is given; but if the denominator n is given, then $\pm D$ is the remainder of the division of An by C , and b the quotient.

Hence we have

$$\frac{C}{A} = \frac{n}{b} \pm \frac{D}{Ab}.$$

This process may be continued in the same manner by making $Dc - Ap = \pm E$, and we have

$$\frac{D}{A} = \frac{p}{c} \pm \frac{E}{Ac},$$

and so on.

5. We next remark, that the number B being less than A by hypothesis, the succeeding numbers C , D , &c. are also less than A , for they are the remainders of the divisions of Ba , Cb , &c. by A ; hence it is obvious that the numerators m , n , p , &c. can never be greater than their respective denominators a , b , c , &c.

But in considering the equation $Ba - Am = \pm C$, if $Ba > Am$, then $Ba - Am = C$; therefore, $Am = Ba - C > Ba$, but A being $> B$, it follows that m is necessarily $<$ than a . If, on the contrary, $Am > Ba$, then $Ba - Am = -C$, therefore,

$Am = Ba + C$, and thence $A(m-1) = Ba + C - A$, but A being $> C$, $C - A$ is a negative number, therefore, $A(m-1) < Ba$, wherefore, B being $< A$, $m-1$ is necessarily $< a$, and, consequently, $m < a + 1$.

We may demonstrate, in the same manner, from the equation $Cb - An = +D$, that in every case $n < b + 1$, and so on.

When the denominators a , b , &c. are given, and the numerators m , n , &c. are to be determined, so that the remainders of the divisions of Ba , Cb , &c. by A may be positive, then it necessarily follows, from the preceding demonstration, that $m < a$, $n < b$, $p < c$, &c.

6. By substituting successively the values of $\frac{C}{A}$, $\frac{D}{A}$, &c. we have this series of transformed expressions,

$$\begin{aligned}\frac{B}{A} &= \frac{m}{a} + \frac{C}{Aa} \\ &= \frac{m}{a} + \frac{n}{ab} + \frac{D}{Aab} \\ &= \frac{m}{a} + \frac{n}{ab} + \frac{p}{abc} + \frac{E}{Aabc}, \text{ \&c.};\end{aligned}$$

where it is to be observed, with respect to the ambiguous sign, that the first is the same as that of the first remainder; that the second must be the product of those of the two first remainders; that the third must be the product of those of the three first remainders, and so on.

These transformations have the advantage of reducing the given fraction to a series of decreasing fractions, the numerators or denominators of which are given, and which approach the nearest possible to the given fraction.

7. If the denominators a , b , c , &c. are supposed given, and equal to each other, then the series takes this more simple form,

$$\frac{B}{A} = \frac{m}{a} + \frac{n}{a^2} + \frac{p}{a^3}, \text{ \&c.}$$

and it is obvious, that if we make $a = 10$, and take the remainders all positive; that is, if we make all the divisions *en defect*, as is done in arithmetic, we shall have the known reduction of the

fraction $\frac{B}{A}$ into decimals, where the numerators m , n , p , &c.

are the successive figures of the fraction. In effect m will be the quotient of the division of Ba , or $10B$ by A , and C the remainder; n will be the quotient of the division of aC , or $10C$ by A , and D the remainder; and so on, which is the known operation

operation of division in decimals. If for a we take the numbers 2, 3, 12, &c. we shall have binary, ternary, duodecimal fractions, &c.

8. I now remark, that when all the denominators are given, and equal, the numerators m , n , p , &c. must necessarily recur to the same values, and form a periodical series; for the remainders C, D, E, &c. being all less than the divisor A, it will necessarily happen, that in the succession of operations, one of the remainders will be repeated. Let us suppose, for example, that the remainder E is equal to the remainder C; that as n is the quotient, and D the remainder of the division of Cb by A; and also, q the quotient, and F the remainder of the division of Ed by A; it follows, because of b and d being each equal to a , that $q = n$ and $F = D$; and for the same reason $n = p$ and $G = E$, and so on; thus the quotients n , p , &c. recur always to infinity, and form a periodic series of two terms. This is what is known to happen in common arithmetic, when any fraction is reduced to decimals. The same thing must consequently take place in any other system of arithmetic.

Hence, reciprocally, we may conclude, that if any numerical series of the form,

$$\frac{m}{a} \pm \frac{n}{a^2} \pm \frac{p}{a^3} \pm, \&c.$$

which goes on to infinity, be of such a nature, that the numerators n , p , &c. (which ought to be all $< a \pm 1$) do not recur in a periodical manner, that series can never represent a rational fraction.

9. When the numerators m , n , p , &c. are the given quantities, and it is required to find the denominators a , b , &c. by the supposed conditions, the numbers A, B, C, D, &c. will necessarily form a decreasing series. For, in the first place, B is $< A$ by hypothesis; and next, C being the remainder of the division of Am by B, will be less than B; in like manner, D being the remainder of the division of Bn by C, will be less than C, and so on; hence it follows that the series of remainders C, D, E, &c.

ought necessarily to terminate in 0, and then the series $\frac{m}{a} \pm \frac{n}{ab}$

$\pm \frac{p}{abc}$, &c. will also terminate, as is evident from the formulas

of No. 6.

Therefore, reciprocally, if we have a series of the form,

$$\frac{m}{a} \pm \frac{n}{ab} \pm \frac{p}{abc} \pm, \&c.$$

where the numerators m , n , p , &c. are respectively less than

(d p)

$a \pm 1$,

$a + 1, b + 1, c + 1, \&c.$; this series, if it goes on to infinity, can never represent a rational fraction; of consequence, it will necessarily represent an irrational numerical quantity.

10. It is known that if e denote the number of which the hyperbolic logarithm is unity, we have always

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3} + \&c.;$$

therefore, if $u = 1$, or $= \frac{1}{i}$, being any whole number what-

ever, the series which represent the value of $e^{\frac{1}{i}}$ will be of the form

we have been considering; hence it follows, that the number $e^{\frac{1}{i}}$ will be necessarily irrational.

We have also, as is known,

$$\text{fin. } u = u - \frac{u^3}{2 \cdot 3} + \frac{u^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

$$\text{cof. } u = 1 - \frac{u^2}{2} + \frac{u^4}{2 \cdot 3 \cdot 4} - \frac{u^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \&c.$$

Therefore, if we make $u = 1$, or $= \frac{1}{i}$, that is, if we take u

equal to the radius, or to any aliquot part of the radius, the series which represents the sine and cosine of that arch have the conditions we have been considering; and as they go on to infinity, we may conclude that the sine or cosine can never be commensurable to the radius.

11. Let us now consider more particularly the case in which the numerators $m, n, \&c.$ are given, and let us suppose them all equal to unity, which will render the series the simplest and the most converging possible.

Therefore, in this case, we are to assume,

$$Ba - A = \pm C, Cb - A = \pm D, Dc - A = \pm E, \&c.$$

and we shall have,

$$\frac{B}{A} = \frac{1}{a} \pm \frac{1}{ab} \pm \frac{1}{abc} \pm, \&c.$$

where, with respect to the ambiguous signs of the series, the rule of No. 6, is to be observed.

Thus we are to take for a the quotient of the division of A by B ; for b the quotient of the division of A by the remainder C , of the preceding division; for c the quotient of the division of A by the remainder D , of the preceding division; and so on. So that

that in these operations all the remainders of the same dividend *A* are compared successively, which renders the series of remainders decreasing; and that of the quotients *a, b, c, &c.* increasing, until at last there be no remainder; and here the operation and the series terminate.

If the divisions are made so as to be all *en defect*, that is, in the ordinary way (No. 3,) the remainders *C, D, &c.* will all have the negative sign; and, therefore, the signs of the series will be alternately positive and negative (No. 6): but that the terms of the series may be all positive, the successive divisions ought to be all *en excess*, in order that the remainders *C, D, &c.* in the preceding formulas may all have the sign $+$. However, if it be required to have the series the most converging possible, the divisions must be made *en defect*, or *en excess*, according as the one or the other of these methods produces the smallest remainder.

12. As this method of converting a fraction into a series is little known, and may be useful in many cases, we shall illustrate it by some examples.

Let the fraction be $\frac{887}{1103}$. Here follows the whole operation,

in which I have performed all the division *en defect*.

$$\begin{array}{r}
 887 \overline{) 1103} \quad 1 \\
 \underline{887} \\
 216 \overline{) 1103} \quad 5 \\
 \underline{1080} \\
 23 \overline{) 1103} \quad 47 \\
 \underline{1081} \\
 22 \overline{) 1103} \quad 50 \\
 \underline{1100} \\
 3 \overline{) 1103} \quad 367 \\
 \underline{1101} \\
 2 \overline{) 1103} \quad 551 \\
 \underline{1102} \\
 1 \overline{) 1103} \quad 1103 \\
 \underline{1103} \\
 0
 \end{array}$$

It appears that the remainders are $-216, -23, -22, -3, -2, -1$; and the quotients $1, 5, 47, 50, 367, 551, 1103$, so that there results this alternate series,

$$\begin{aligned}
 \frac{887}{1103} &= 0 - \frac{1}{5} + \frac{1}{5 \cdot 47} - \frac{1}{5 \cdot 47 \cdot 50} + \frac{1}{5 \cdot 47 \cdot 50 \cdot 367} \\
 &\quad - \frac{1}{5 \cdot 47 \cdot 50 \cdot 367 \cdot 551} + \frac{1}{5 \cdot 47 \cdot 50 \cdot 367 \cdot 551 \cdot 1103}
 \end{aligned}$$

Let

Let us take the fraction which expresses the ratio of the circumference of a circle to its diameter, and which is, in decimals, $3.14159265358979323846264338$, &c.

By performing the same operation upon the fraction

$$\frac{141592653589793238462643 \dots}{1000000000000000000000000 \dots}$$

and making the divisions *en defect*, or *en excess*, according as is found necessary to produce each remainder less than the half of that which preceded it, we shall have the ratio under consideration equal to this quickly converging series

$$3 + \frac{1}{7} - \frac{1}{7 \cdot 113} - \frac{1}{7 \cdot 113 \cdot 4739} + \frac{1}{7 \cdot 113 \cdot 4739 \cdot 47051} \\ + \frac{1}{7 \cdot 113 \cdot 4739 \cdot 47051 \cdot 499762} - \&c.$$

The two first terms when united give the known proportion of Archimedes, viz. $\frac{22}{7}$; and by adding the third term, we have the proportion of Metius, or $\frac{355}{113}$.

13. In the preceding problems it has been considered how a given fraction may be reduced to other fractions, the numerators or denominators of which are given; but it may be only required to reduce a fraction to other fractions expressed in the least terms, and which shall be the nearest possible to the given fraction. As this problem is the most interesting in arithmetic, both on account of the artifices which it requires, and the purposes to which it may be applied, we proceed to give here a solution of it, deduced from the same principles.

14. According to the formulas of No. 4, we have this transformation $\frac{B}{A} = \frac{m}{a} \pm \frac{C}{\Lambda a}$, or $Ba - Am = \pm C$. But when m and a are indeterminate, and that it is required to determine them, such, that the fraction $\frac{m}{a}$ shall approach the nearest possible to the given

fraction $\frac{B}{A}$, the numbers m and a , being respectively less than the numbers B and A , it is evident that we must give to C the least value possible, that is, we must make $C = \pm 1$, for the supposition of $C = 0$ would imply the perfect equality of the two fractions, as then we should have $m = B$, $a = A$.

Therefore, m and a are to be such that $Ba - Am = \pm 1$,
hence

hence we have this transformation,

$$\frac{B}{A} = \frac{m}{a} \pm \frac{1}{Aa},$$

where the upper or lower sign is to be taken according as the given fraction is greater or less than the new fraction $\frac{m}{a}$.

15. But it is necessary to be first assured that two numbers $a < A$, $m < B$ may be found, such that $Ba - Am = \pm 1$.

Let us put this equation under the form $a = \frac{Am + 1}{B}$; then, that

this may be possible, it is evident that we must find a number m less than B , which shall render the number $m A \pm 1$ divisible by B . Now, if in $m A \pm 1$ there be substituted successively for m , all the numbers $0, 1, 2$, &c. to $B - 1$, and each result be divided by B , the remainders will be all less than B , and different from each other; for if there can be two equal remainders, let m and m' be the two numbers which give the same remainder, then the difference $(m - m') A$ will be divisible by B , but A and B are prime to each other, and $m - m'$ is a number less than B , seeing that m and m' are less than B , therefore this difference cannot be divisible by B ; hence it follows, that the two remainders cannot be equal, therefore 0 must necessarily be found among the remainders; and, consequently, there may be always a number less than B , which substituted for m in $m A \pm 1$ shall render this last number divisible by B . The same number may, therefore, be taken for m , and the quotient of the division of $A m \pm 1$ by B will be the corresponding value of a , which will, consequently, be less than A .

It appears also by this demonstration that there can only be one value of m and one of a , less than B and A , which will satisfy the equation $Ba - Am = \pm 1$. For let a and a' be two values of a , and m , m' two values of m , we have, therefore, $Ba - Am = \pm 1$, $Ba' - Am' = \pm 1$, and, by subtracting the one equation from the other, $B(a - a') = A(m - m')$; but this equation cannot take place in whole numbers, for A and B are prime to each other, and $a - a'$ and $m - m'$ are numbers less than A and B . Therefore, two numbers m and a may always be found so as to satisfy the equation, by repeated trials of numbers less than B or A for m or a ; but we shall afterwards give direct methods for finding them.

16. Again, when two values of m and a have been found so as to satisfy the equation $Ba - Am = \pm 1$, if we take $a' = A - a$, $m' = B - m$, we shall have $Ba' - Am' = \pm 1$, thus it appears that A is sufficient, if such values of m and a be found as to satisfy the proposed equation, when the ambiguous sign is taken either as positive

positive or as negative at pleasure. It also appears that it is always possible to give to a or m values greater or less than $\frac{A}{2}$ or $\frac{B}{2}$, for it is obvious that if a is $> \frac{1}{2} A$, $A - a$ will be $< \frac{1}{2} A$.

But the fraction $\frac{m}{a}$ approaches so much the nearer to the fraction $\frac{B}{A}$, whether the difference be in excess or defect, according as the denominator a is greater, for without considering the sign the difference is $\frac{1}{Aa}$, therefore, if we take $a > \frac{A}{2}$ that difference is the least, and $< \frac{2}{Aa}$.

17. It further appears from the equation $Ba - Am = \pm 1$, which is to be satisfied by the numbers m and a , 1^o that these numbers are prime to each other, so that the fraction $\frac{m}{a}$ is reduced to its lowest terms, for if they had any other divisor than unity, the same would divide the second member of the equation, which is impossible, and 2^o that it is impossible for any other fraction to fall between the two fractions $\frac{B}{A}$ and $\frac{m}{a}$, except that it has a denominator greater than A , for suppose that there may be such a fraction, as $\frac{\mu}{\alpha}$, whose value falls between the two fractions, and of which the denominator α is $< A$, then the difference between the two fractions $\frac{\mu}{\alpha}$ and $\frac{m}{a}$, will be less than the difference between the fractions $\frac{B}{A}$ and $\frac{m}{a}$; now the first of these differences is $\frac{\mu a - \alpha m}{\alpha a}$, and the second is $\pm \frac{1}{Aa}$; but it is evident that the number $\mu a - \alpha m$ cannot be less than unity, and as $\alpha < A$ by hypothesis, it follows, on the contrary, that the first difference is always greater than the second difference.

18. If the fraction $\frac{m}{a}$ is also expressed in too high terms, it may be reduced in the same manner; seeing that the numbers m and a are prime to each other. Therefore two numbers n and b are to be sought

found, which are less than m and a , so as to satisfy the equation $m b - a n = \pm 1$, and then

$$\frac{m}{a} = \frac{n}{b} \pm \frac{1}{ab}.$$

In like manner, if two numbers p and c are found which are less than n and b , and such, that $nc - bp = \pm 1$, we have also

$$\frac{n}{b} = \frac{p}{c} \pm \frac{1}{bc};$$

and so on.

These new fractions $\frac{n}{b}$, $\frac{p}{c}$, &c. are also reduced to their lowest terms, and are expressed always by the smallest numbers in such a manner between any two consecutive fractions of the series $\frac{B}{A}$, $\frac{m}{a}$, $\frac{n}{b}$, $\frac{p}{c}$, &c. there can fall no other fraction of which the denominator is also between the denominators of the two fractions, (No. 17.). Hence it follows, that this series of fractions contains all the fractions which being successively expressed by smaller numbers than the fraction $\frac{B}{A}$ approach nearer to it than any other fraction can; which is not expressed by greater numbers.

19. Let us suppose that the numbers a, b, c, d , &c. m, n, p, q , &c., are taken, so that the upper signs or the lower signs always take place in the equations

$$Ba - Am = \pm 1, mb - an = \mp 1, nc - bp = \pm 1, pd - Cq = \mp 1, qc - dr = \pm 1, \&c.,$$

then, by preserving the same law of the signs, we have these approximations,

$$\frac{B}{A} = \frac{m}{a} \pm \frac{1}{Aa}, \quad \frac{m}{a} = \frac{n}{b} \pm \frac{1}{ab}, \quad \frac{n}{b} = \frac{p}{c} \pm \frac{1}{bc};$$

$$\frac{p}{c} = \frac{q}{d} \mp \frac{1}{cd}, \quad \frac{q}{d} = \frac{r}{e} \pm \frac{1}{de}, \quad \&c.$$

which it appears are alternately in excess and in defect.

If the values of $\frac{m}{a}$, $\frac{n}{b}$ be successively substituted, we shall

have

$$\begin{aligned}\frac{B}{A} &= \frac{m}{a} \pm \frac{1}{Aa} \\ &= \frac{n}{b} \pm \left(\frac{1}{Aa} - \frac{1}{ab} \right) \\ &= \frac{p}{c} \pm \left(\frac{1}{Aa} - \frac{1}{ab} + \frac{1}{bc} \right) \\ &= \frac{q}{d} \pm \left(\frac{1}{Aa} - \frac{1}{ab} + \frac{1}{bc} - \frac{1}{cd} \right) \&c.\end{aligned}$$

But as these numbers are supposed to go on decreasing, it is evident that $\frac{1}{ab} > \frac{1}{Aa}$, $\frac{1}{bc} > \frac{1}{ab}$, $\frac{1}{cd} > \frac{1}{bc}$, &c. Therefore, we have, in consequence of the preceding formulas,

$$\begin{aligned}\frac{B}{A} &> \frac{m}{a} > \frac{p}{c} > \frac{r}{e}, \&c. \\ &> \frac{n}{b} > \frac{q}{d} > \frac{s}{f}, \&c.\end{aligned}$$

The upper signs correspond to the upper signs of the formulas, and the lower to the lower.

Whence, I conclude, that if it be supposed, that

$$Bb - An = \pm N, Bc - Ap = \pm P, Bd - Aq = \mp Q, Bc - Ar = \pm R, Bf - As = \mp S, \&c.$$

(preserving always the upper or the lower signs,) the numbers N, P, Q, &c. shall be necessarily all positive, and it is evident that we shall have

$$\begin{aligned}\frac{B}{A} &= \frac{m}{a} \pm \frac{1}{Aa} \\ &= \frac{n}{b} \mp \frac{N}{Ab} \\ &= \frac{p}{c} \pm \frac{P}{Ac} \\ &= \frac{q}{d} \mp \frac{Q}{Ad}.\end{aligned}$$

20. This being premised, if the two equations $Ba - Am = \pm 1$, and $mb - an = \mp 1$ be added together, we have $(B - n)a - (A - b)m = 0$, that is $\frac{B - n}{A - b} = \frac{m}{a}$. But the fraction $\frac{m}{a}$ being reduced to its lowest terms, the fraction $\frac{B - n}{A - b}$ cannot be equal to it unless the numerator $B - n$ and the denominator $A - b$

$A - b$ are equimultiples of m and a ; we have therefore, taking for λ a whole indeterminate number, $B - n = \lambda m$, $A - b = \lambda a$; therefore, $n = B - \lambda m$, $b = A - \lambda a$.

But as n ought to be $< m$, and $b < a$, it is evident that we can only take for n and b the remainders of the divisions of B by m , and of A by a , and λ will then be the common quotient of these

divisions. Thus, knowing the two first fractions $\frac{B}{A}, \frac{m}{a}$, we may find,

in this manner, the third $\frac{n}{b}$. In like manner the equations

$mb - an = \mp 1$, and $nc - bp = \pm 1$ being added together

give $(m - p)b - (a - c)n = 0$, that is $\frac{m - p}{a - c} = \frac{n}{b}$; hence

we find as before $m - p = \mu n$, $a - c = \mu b$, μ being any whole number. We have thus $p = m - \mu n$, $c = a - \mu b$, and as p ought to be $< n$, and $c < b$, it follows that p and c can only be the remainders of the divisions of m by n , and of a by b , and that μ is their common quotient.

In the same way, from the two equations $nc - bp = \pm 1$, $pd - cq = \mp 1$, we derive these two formulas $q = n - \nu p$, $d = b - \nu c$, ν being a whole indeterminate number. And as q and d ought to be respectively less than p and c , we thence conclude that they can only be the remainders of the divisions of n by p , and of b by c , and that ν is their common quotient, and so on.

21. We have found the formulas

$$n = B - \lambda m, p = m - \mu n, q = n - \nu p, \&c.$$

$$b = A - \lambda a, c = a - \mu b, d = b - \nu c, \&c.$$

If these values be substituted in the expressions of the numbers $N, P, Q, \&c.$ of No. 19, it will be found, because of $Ba - Am = \pm 1$, and by taking away the ambiguous sign which affects all the terms, that

$$N = \lambda, P = 1 + \mu N, Q = N + \nu P, R = P + \pi Q, S = Q + \rho R, \&c.$$

these formulas shew that the numbers $N, P, Q, \&c.$ necessarily go on increasing, while the numbers $a, b, c, d, \&c.$ go on decreasing. These formulas may also serve to determine directly the value of the numbers when the coefficients $\lambda, \mu, \nu, \&c.$ are known, and the same numbers $N, P, Q, \&c.$ serve to express in

a simple manner the differences of the fractions $\frac{m}{a}, \frac{n}{b}, \&c.$ and of the fraction $\frac{B}{A}$, (No. 19).

22. Now as the numbers $m, n, p, \&c.$ also the numbers $a, b, c, \&c.$ must go on decreasing, it is evident, that by continuing

tinuing the same operations, we shall at last come to terms which are $\equiv 0$.

Let us suppose for example that $f \equiv 0$, then the equation $rf - es \equiv \mp 1$ (No. 19) becomes $-es \equiv \mp 1$, therefore we must take the upper sign and make $e \equiv 1$, $s \equiv 1$. Therefore the equation $Bf - As \equiv \mp S$ (of the same number) becomes by taking the upper sign $-A \equiv -S$, that is $S \equiv A$. Again, the equation which precedes it, $Be - Ar \equiv \pm R$, gives $B - Ar \equiv R$: therefore, since $B < A$, in order that R may be positive, we must take $r \equiv 0$, and hence $R \equiv B$.

23. Having thus the values of the two last terms of the series N, P, Q , &c.; we may find the values of all the preceding, as well as those of the numbers λ, μ , &c. by the formulas of No. 21, $\lambda \equiv N$, $1 \equiv P - \mu N$, $N \equiv Q - \nu P$, $P \equiv R - \pi Q$, $Q \equiv S - \rho R$, &c.

Having now found $S \equiv A$ and $R \equiv B$ we have $Q \equiv A - \rho B$, but Q must be $< R$, by the equation $P \equiv R - \pi Q$, therefore Q being $< B$, it is evident that Q must be the remainder of the division of R by Q , and that π will be its quotient, hence we have Q .

Again the equation $P \equiv R - \pi Q \equiv B - \pi Q$ shews in like manner (because that P must be $< Q$ by the equation $N \equiv Q - \nu P$) that P must be the remainder of the division of R by Q , and that π is the quotient.

In like manner the equation $N \equiv Q - \nu P$ (in which N must be $< P$ because of the equation $1 \equiv P - \mu N$ which precedes it,) shews that N must be the remainder of the division of Q by P , and that ν must be the quotient.

Finally, the equation $1 \equiv P - \mu N$ gives $\mu \equiv \frac{P-1}{N}$, and the equation $\lambda \equiv N$ gives the value of λ .

24. The number λ, μ, ν, π , being thus known, we can find directly the numbers m, n, p , &c. and a, b, c, d , &c., by the formulas of No. 21. In effect these formulas give

$$m \equiv \mu n + p, n \equiv \nu p + q, p \equiv \pi q + r, q \equiv \rho r + S, \&c.$$

$$a \equiv \mu b + c, b \equiv \nu c + d, c \equiv \pi d + e, d \equiv \rho e + f, \&c.$$

and as it has been found (No. 22) that $f \equiv 0$, $e \equiv 1$, $s \equiv 1$, $r \equiv 0$, we have by going back, successively, the values d, c, b, a , and of q, p, n, m .

But all these operations may be performed at once, as will appear from the following example.

25. Let the proposed fraction be $\frac{887}{1103}$, so that we have $A \equiv 1103$ and $B \equiv 887$; the calculation is disposed thus :

1 A —

1 A —	0 B =	1103	
0 A —	1 B =	— 887	1
1 A —	1 B =	216	4
4 A —	5 B =	— 23	9
37 A —	46 B =	9	2
78 A —	97 B =	— 5	1
115 A —	143 B =	4	1
193 A —	240 B =	— 1	4
887 A —	1103 B =	0.	

The two first equations are always $1A - 0B = A$, and $0A - 1B = -B$. To find a third from them, we seek how often B is contained in A ; here it is contained once, and 1 is put by the side of the second equation; afterwards we add this equation multiplied by the same number 1 to the preceding equation, and thus obtain the third equation $1A - 1B = 216$; we again enquire how often the number 218 is contained in the preceding number 887 , and find that it is contained 4 times, thus we put 4 to the side of this equation and the product of this equation by 4 , added to the preceding, gives the following $4A - 5B = -23$, and so on.

It immediately appears from this process, that the numbers of the third column are the remainders, and these of the fourth the quotients of the different divisions which take place in the known operation for finding the greatest common divisor of the two first numbers of the third column; now these numbers being supposed prime to each other, it follows that we must necessarily come at last to a remainder equal to unity, which produces directly an equation of this form $mA - aB = \pm 1$, so that we have necessarily in this manner a solution of that equation, and, consequently, proper values of m and a , from whence we may derive successively these of n, p, q , &c. and of b, c, d , &c. by the method of No. 20.

26. But in comparing these operations with those to which our analysis has formerly conducted us (No. 23), it is easy to see that the coefficients of A , in the first column, are all the numbers m, n, p , &c.; and the coefficients of B in the second column are the corresponding numbers a, b, c , &c.; these numbers being disposed in a contrary order, so that the first m and a are the last but one, namely 193 and 240 . It likewise appears that the numbers which form the third column are the numbers $1, N, P, Q$, &c. disposed also in a contrary order, and taken alternately in excess and in defect; so that we have $N = 4, P = 5, Q = 9$, &c. Finally, the numbers of the fourth column are the numbers λ, μ, ν , &c. taken also in a contrary order, so that we have $\lambda = 4, \mu = 1, \nu = 1$, &c. Thus we have immediately the fractions which are nearest to the given fraction, and expressed in the smallest terms, by supposing the third column equal to q , and taking successively for

for $\frac{B}{A}$, the values which result from this supposition.

In the case of the preceding example, where the given fraction is $\frac{887}{1103}$, the converging fractions are therefore $\frac{193}{240}$, $\frac{115}{143}$, $\frac{78}{97}$, $\frac{37}{46}$,

$\frac{4}{5}$, $\frac{1}{1}$, $\frac{0}{1}$; and the equations $193A - 240B = -1$, $115A - 143B = 4$, &c., make it appear that the error of the first of these

fractions is in excess, that of the second in defect, and so on.

If it be required to have the first fraction in defect we must take (No. 16), the fraction $\frac{887 - 193}{1103 - 240}$, that is $\frac{694}{863}$; and if it be required to have the second fraction in excess we must take the fraction $\frac{193 - 115}{240 - 143}$, that is, $\frac{78}{97}$; and so of others.

27. Again, if the formulas of No. 21, are put under the form

$$\frac{P}{N} = \mu + \frac{1}{\lambda}, \quad \frac{Q}{P} = \nu + \frac{1}{P}, \quad \frac{R}{Q} = \pi + \frac{1}{Q}, \quad \frac{S}{R} = \rho + \frac{1}{R}, \quad \&c.$$

we shall have by successive substitutions

$$\frac{P}{N} = \mu + \frac{1}{\lambda}$$

$$\frac{Q}{P} = \nu + \frac{1}{\mu + \frac{1}{\lambda}}$$

$$\frac{R}{Q} = \pi + \frac{1}{\nu + \frac{1}{\mu + \frac{1}{\lambda}}}$$

$$\frac{S}{R} = \rho + \frac{1}{\pi + \frac{1}{\nu + \frac{1}{\mu + \frac{1}{\lambda}}}}$$

&c.

So that as the two first terms of the series N, P, Q , &c. are equal to B , and A , (No. 22), it follows that the fraction $\frac{B}{A}$ is thus reduced to a continued fraction, the successive denominators of which are the numbers λ, μ, ν , &c. taken in an inverted order, and, therefore, the very number of the fifth column in the table of No. 25.

Thus

Thus the fraction $\frac{887}{1103}$ (the example of that number) is reduced to the continued fraction

$$\begin{array}{c} \frac{1}{1 + \frac{1}{4 + \frac{1}{9 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4}}}}}}} \end{array}$$

and this fraction being successively carried on no further than the first denominator, the second, the third, &c. gives the same converging fractions as those already found, viz.

$$\frac{1}{1}, \frac{4}{5}, \frac{37}{46}, \text{ \&c.}$$

28. The problem of finding fractions which converge the most possible to a given fraction, has been usually resolved by means of continued fractions. The first step is to reduce that fraction to a continued fraction, by performing on the numerator and denominator the known operation for finding their greatest common divisor; and, taking the quotients of the successive divisions for the denominators of the continued fraction; the converging fractions are thence found by formulas similar to those of No. 24.

The fraction $\frac{887}{1103}$ treated in this way gives the quotients 1,

4, 9, 2, 1, 1, 4, and thence is formed this series of fractions,

$$\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{4}{5}, \frac{37}{46}, \frac{78}{97}, \frac{115}{143}, \frac{193}{240}, \frac{887}{1103},$$

every one of which is formed from the two preceding, by taking for the numerator the sum of the preceding numerator multiplied by the number of the series of quotients which is placed above it, and of the numerator which precedes that one; and for the denominator, the sum of the preceding denominator, multiplied by the same number, and of the denominator which precedes this last.

29. But another method may be deduced from our formulas, by which we can find in a direct manner any one of the converging fractions by means of one series alone. For this purpose I conclude the equations of No. 21, under this form,

$$B = \lambda m + n, m = \mu, n = \nu p + q, \text{ \&c.}$$

$$A = \lambda a + b, a = \mu b + c, b = \nu c + d, \text{ \&c.}$$

and I observe that the values of B and A depend on the quantities μ, ν, π , &c., and those of n and b depend on ν, π , &c., which is evident from the very form of the equations. On the other hand, if we consider the series of numbers N, P, Q, &c.

to

to which there is in the first place joined the two numbers $L = 0$ and $M = 1$, we have by the formulas of No. 24, $N = \lambda M + L$, $P = \mu N + M$, $Q = \nu P + N$, $R = \pi Q + P$, &c. but it has appeared (No. 22) that the two last terms of this series are necessarily the numbers B and A ; therefore, if in these formulas we change the numbers λ, μ, ν , &c. into μ, ν, π , &c., the two last terms of the series will be m and a , so that we have immediately the first converging fraction $\frac{m}{a}$ by the two last terms of the series $L = 0$, $M = 1$, $N = \mu M + L$, $P = \nu N + M$, $R = \pi P + N$, &c., and for the same reason we have the second converging fraction $\frac{n}{b}$, by taking for n and b the two last terms of the series $L = 0$, $M = 1$, $N = \nu M + L$, $P = \pi M + MN$, &c. and so on: the numbers λ, μ, ν , being the denominators of the continued fraction taken in a retrograde order, that is to say, beginning with the last.

30. Thus having found for the fraction $\frac{887}{1103}$ this series of quotients or denominators 1, 4, 9, 2, 1, 1, 4, we may form the following series

$$0, \overset{4}{1}, \overset{1}{4}, \overset{1}{5}, \overset{2}{9}, \overset{9}{23}, \overset{4}{216}, \overset{1}{887}, 1103$$

$$0, \overset{1}{1}, \overset{1}{1}, \overset{2}{2}, \overset{9}{5}, \overset{4}{47}, \overset{1}{193}, 240$$

$$0, \overset{1}{1}, \overset{2}{1}, \overset{9}{3}, \overset{4}{28}, \overset{1}{115}, 143, \&c.$$

where every term is composed of the preceding term, multiplied by the number which is above it, and of that which precedes it; and it appears that the first series reproduces the two terms of the proposed fraction, that the second gives the two terms of the first converging fraction $\frac{193}{240}$, that the third series gives the

terms of the second converging fraction $\frac{115}{143}$, and so on. The numbers placed above the terms of these series are, as it appears, the denominators of the continued fraction written in their order, beginning at the last, or the last but one, or the last but two, &c.

31. We have now considered the problem of the reduction of fractions to others which are more simple in a general manner; and we have derived from the same principle the theory of decimal fractions considered according to any system of numeration; the theory of another species of fractions little known, and first proposed, I believe by the late Lambert, which possesses the singular advantage of forming a series more converging than any geometrical series; in short the theory of continued fractions which has hitherto been always treated in an isolated manner. The only object of this memoir has been to shew how these different theories may be compared together, and presented under one and the same point of view.

ARTICLE III.

The Inverse Method of Central Forces.

BY MR. JOHN DAWSON*.

PROPOSITION I. Fig. 152. Pl. 8.

THE centripetal force being inversely as the n th power of the distance from the centre, and the direction and velocity of a body at any point V being given; to determine the orbit, &c.

Let C be the centre of force, V the point from which the body is projected, VnW the trajectory in which it moves, and VZ a circle described from the centre C at the distance CV: to the points V, n, of the orbit, draw the tangents VP, np; and from the centre of force C, let fall the perpendiculars CP, Cp, upon the tangents; join C, n, and produce Cn to Y, and draw CtX indefinitely near to CnY; lastly, draw tv perpendicular to Cn. Put CV = r , CP = P , Cn = y , Cp = p , the velocity at V = v , the velocity at n = v' , the velocities being measured by the spaces described in the time (1).

If O be the centre of curvature of the trajectory at the point n, then it is well known that $On = \frac{y\dot{y}}{p}$. Draw OR perpendicular to Cn; the triangles Cpn, ORn, are similar, therefore

$Cn(y) : Cp(p) :: On\left(\frac{y\dot{y}}{p}\right) : Rn = \frac{p\dot{y}}{p} = \text{half the chord of curvature passing through the centre of force. But the centripetal force estimated by the velocity generated in the time (1), is = the square of the velocity divided by } \frac{1}{2} \text{ the chord of curvature}$
 $= v'^2 \div \frac{p\dot{y}}{p} = \frac{v'^2 \dot{p}}{p\dot{y}}$; and $p^2 : P^2 :: v^2 : v'^2$; therefore the centripetal force = $\frac{P^2 v^2 \dot{p}}{p^3 \dot{y}}$.

* From the Manchester Memoirs, Vol. 4 and 5.

Let this expression be put $= \frac{A^{n+1}}{y^n}$, A, being a constant quantity; wherefore

$$\frac{P^2 v^2 \dot{p}}{\dot{p}^2 \dot{y}} = \frac{A^{n+1}}{y^n}.$$

Multiply both sides of the equation by $\dot{y}$, and take the fluents, then

$$\frac{P^2 v^2}{2 \dot{p}^2} = \frac{A^{n+1}}{(n-1) y^{n-1}}.$$

But at V this equation becomes $\frac{P^2 v^2}{2 P^2} = \frac{A^{n+1}}{(n-1) r^{n-1}}$, therefore the correct fluent gives

$$\frac{P^2 v^2}{2 \dot{p}^2} - \frac{v^2}{2} = \frac{A^{n+1}}{(n-1) r^{n-1}} \times \frac{r^{n-1} - y^{n-1}}{y^{n-1}}.$$

If a body descend directly towards the centre by an accelerating force $= \frac{A^{n+1}}{y^n}$, and u = the velocity it has acquired when at the distance y from that centre, then it is well known, that

$$\frac{A^{n+1}}{y^n} \times -\dot{y} = u \dot{u}. \text{ Take the fluents; then } \frac{A^{n+1}}{(n-1) y^{n-1}} = \frac{u^2}{2},$$

or $\frac{2A^{n+1}}{(n-1) y^{n-1}} = u^2$, which requires no correction if the body descend from an infinite height; for in that case both sides of the equation vanish at the same time. At the point V, therefore, or at the distance r , from the centre, the square of the velocity acquired by descending from an infinite height $= \frac{2A^{n+1}}{(n-1) r^{n-1}}$.

Let therefore in general $\frac{2mA^{n+1}}{(n-1) r^{n-1}} = v^2$, where m may be either greater, equal to, or less than unity. Hence by substitution and reduction, we have

$$1. \quad \dot{p} = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times P y^{\frac{n-1}{2}}}}{\sqrt{\left(y^{n-1} + \frac{r^{n-1}}{m-1}\right)}}, \quad m > 1.$$

$$2. \quad \dot{p} = \frac{P}{r^{\frac{n-1}{2}}} \times y^{\frac{n-1}{2}}, \quad m = 1.$$

$$3. \quad \dot{p} =$$

$$3 \quad p = \frac{\sqrt{\left(\frac{m}{1-m}\right) \times P y^{\frac{n-1}{2}}}}{\sqrt{\left(\frac{r^{n-1}}{1-m} - y^{n-1}\right)}} \quad m < 1.$$

Therefore it is evident, that if a body be acted upon by a single force which tends to a fixed point, the law of that force being given, the equation of the curve it describes has been determined: except in that particular case where $n = 1$, or the force is inversely as the distance of the body from the centre; for then the above equations fail.

Take $s : 1 ::$ velocity in the curve at $V(v) : \text{velocity in a circle at the same distance} = \frac{v}{s}$,

$$\text{then } \frac{v^2}{s^2} \div r = \frac{v^2}{s^2 r} = \text{force at } V = \frac{A^n + 1}{r^n};$$

$$\text{whence } v^2 = \frac{s^2 A^n + 1}{r^n - 1} = \frac{2m \times A^n + 1}{(n-1)r^{n-1}};$$

therefore $s^2 = \frac{2m}{n-1}$, and $m = s^2 \times \frac{n-1}{2}$; which value of m may be substituted for it in any of the three equations found above.

When $n = 1$, we have $\frac{P^2 v^2 \dot{p}}{p^3} = \frac{A^2 \dot{y}}{y}$; hence, by taking the fluents, $\frac{P^2 v^2}{2 P^2} = A^2 \times \log. \frac{1}{y}$; but at V , this equation becomes $\frac{P^2 v^2}{2 P^2} = \frac{v^2}{2} = A^2 \times \log. \frac{1}{r}$; therefore the fluent, corrected gives $\frac{P^2 v^2}{2 p^2} - \frac{v^2}{2} = A^2 \times \log. \frac{r}{y}$.

But we have just found that $\frac{v^2}{s^2 r} = \left(\frac{A^n + 1}{r^n} = \right) \frac{A^2}{r}$ upon the present supposition; therefore $A^2 = \frac{v^2}{s^2}$: substitute and reduce, then

$$p^2 = \frac{P^2}{1 + \frac{2}{s^2} \times \log. \frac{r}{y}}, \text{ or } p = \frac{P}{\sqrt{\left(1 + \frac{2}{s^2} \times \log. \frac{r}{y}\right)}}, \text{ the equation of the curve.}$$

Next, to find the angle described, or the position of the line joining the centre of force, and the body, in respect of the line CV for any given value of it. By similar triangles, we have

$$y^2 : p^2 :: (tn)^2 : (tv)^2, \text{ and by division,}$$

$$y^2 - p^2 : p^2 :: (vn)^2 : (tv)^2, \text{ also,}$$

$$y^2 : r^2 :: (tv)^2 : (XY)^2; \text{ ex aq.}$$

$$y^4 - p^2 y^2 : r^2 p^2 :: (vn)^2 : (XY)^2 :: \dot{y}^2 : \dot{z}^2 (\dot{z} \text{ being supposed} \\ = XY); \text{ consequently,}$$

$$\dot{z} = \frac{r p \dot{y}}{y \sqrt{(y^2 - p^2)}}, \text{ whatever be the value of } n.$$

Substitute, therefore, in this equation, the several values of p found above, and there will arise the following equations:

$$1. \dot{z} = \pm \frac{r \sqrt{\left(\frac{m}{m-1}\right)} \times P y^{\frac{n-3}{2}} \dot{y}}{y \sqrt{\left(y^{n-1} - \frac{m}{m-1} \times P^2 y^{\frac{n-3}{2}} + \frac{r^{n-1}}{m-1}\right)}}, \quad m >$$

$$2. \dot{z} = \pm \frac{r P y^{\frac{n-3}{2}} \dot{y}}{\sqrt{(r^{n-1} y^2 - P^2 y^{n-1})}} = \pm \frac{r y^{\frac{n-5}{2}} \dot{y}}{\sqrt{\left(\frac{r^{n-1}}{P^2} - y^{n-3}\right)}}, \quad m = 1.$$

$$3. \dot{z} = \pm \frac{r \sqrt{\left(\frac{m}{1-m}\right)} \times P y^{\frac{n-3}{2}} \dot{y}}{y \sqrt{\left(\frac{r^{n-1}}{1-m} - \frac{m}{1-m} \times P^2 y^{\frac{n-3}{2}} - y^{n-1}\right)}}, \quad m < 1.$$

If $n = 1$, then substituting, in the same equation, the value of p found above, we have

$$\dot{z} = \pm \frac{r P \dot{y}}{y \sqrt{\left\{ \left(1 + \frac{2}{s^2} \times \log. \frac{r}{y}\right) \times y^2 - P^2 \right\}}}. \text{ Hence all the}$$

curves may be constructed.

It must be observed, that the positive or negative sign takes place, according as the body ascends from, or descends towards the centre.

$$\text{Because } p^2 = \frac{m}{m-1} \times P^2 y^{n-1} \text{ in general, and at an}$$

$$y^{n-1} + \frac{r^{n-1}}{m-1}$$

apex it is evident, that $p = y$; hence in that case there arises, by reduction, the following equation; viz.

$$y^{n-1}$$

$y^{n-1} - \frac{m}{m-1} P^2 y^{n-3} + \frac{r^{n-1}}{m-1} = 0$; from whence, and the proper equation for $\dot{z}$, the number and position of the apses may be determined*.

It may be observed that the quantity under the radical in the value of $\dot{z}$ being put $= 0$, gives the equation for determining the apses.

If m be supposed greater than unity, and y infinite, then will $p = \sqrt{\left(\frac{m}{m-1}\right)} \times P =$ a perpendicular upon the asymptote to the curve described by the body, or its distance from the centre of force. This expression, it may be observed, is not affected by the value of n . Hence if the value of z be determined upon the above supposition, the asymptotes to the trajectory may easily be drawn.

If m be less than unity, and y infinite, then $p = \frac{\sqrt{\left(\frac{m}{1-m}\right)} \times P}{\sqrt{-1}}$; therefore the curve has not an asymptote. Lastly, if $m = 1$, the perpendicular upon the asymptote will be infinite.

In all these cases the body may descend to the centre, if n be greater than unity; because in that case, when $y = 0$, $p = 0$. But if n be less than unity, the body can only arrive at the centre in a straight line. For the equation

$$p = \frac{\sqrt{\left(\frac{m}{m-1}\right)} \times P y^{\frac{n-1}{2}}}{\sqrt{\left(y^{n-1} + \frac{r^{n-1}}{m-1}\right)}}, \text{ becomes } p = \frac{m^{\frac{1}{2}} P r^{\frac{1-n}{2}}}{\sqrt{\left\{(m-1)r^{1-n} + y^{1-n}\right\}}};$$

in which case, when $y = 0$, the value of p is positive, infinite, or impossible, unless P at the same time be equal to nothing. But it is evident that p can never be greater than y .

It will likewise easily appear, that when $n = 1$, the body may descend to the centre; for $p^2 = \frac{P^2}{1 + \frac{2}{s^2} \times \log. \frac{r}{y}} = P^2 \div$

by an infinite quantity when $y = 0$.

But if the body begins to ascend, the greatest height at which it can arrive may be found, by making $p = y$; at which time it comes to an apse.

* See the Note placed at the end of this Article.

COR. 1. If $n = 0$, or the force be the same at all distances from the centre, then

$$p = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times P y^{-\frac{1}{2}}}}{\sqrt{\left(y^{-1} + \frac{r^{-1}}{m-1}\right)}}; \text{ but } m = s^2 \times \frac{n-1}{2} = -\frac{s^2}{2};$$

therefore, by substitution, $p = \frac{s P r^{\frac{1}{2}}}{\sqrt{\{(s^2 + 2) \times r - 2y\}}}$ the equation of the curve described by the body.

Likewise, it will easily appear, that,

$$\dot{z} = \frac{s P r^{\frac{3}{2}} \dot{y}}{y \sqrt{\left(-y^3 + \frac{s^2 + 2}{2} \times r y^2 - \frac{s^2}{2} \times P^2 r\right)}}. \text{ The equation}$$

for determining the apsidal distances will be found to be $y^3 - \frac{s^2 + 2}{2}$

$\times r y^2 + \frac{s^2}{2} \times P^2 r = 0$. This equation has three real roots,

as is evident from the latter part of the note to the proposition: two of them are positive and one negative; for the last term of the equation is positive. The negative root belongs to a part of the algebraical curve, whose concavity is turned from the centre of force.

The curve, whose equation is given in this corollary, is that which a particle of light describes in passing through the atmosphere; and by the help of this equation, together with a single observation of the refraction at a given altitude, the curve may be described.

COR. 2. If $n = 2$, or the force be reciprocally as the square of the distance, the value of p will become

$$1. p = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times P y}}{\sqrt{\left(y^2 + \frac{r}{m-1} \times y\right)}} \dots \dots \dots m > 1.$$

$$2. p = \frac{P}{r^{\frac{1}{2}}} \times y^{\frac{1}{2}} \dots \dots \dots m = 1.$$

$$3. p = \frac{\sqrt{\left(\frac{m}{1-m}\right) \times P y}}{\sqrt{\left(\frac{r}{1-m} y - y^2\right)}} \dots \dots \dots m < 1.$$

And

And the respective values of $\dot{z}$ will be

$$1. \dot{z} = \pm \frac{r\sqrt{\left(\frac{m}{m-1}\right) \times P\dot{y}}}{y\sqrt{\left(y^2 + \frac{r}{m-1} \times y - \frac{m}{m-1} P^2\right)}}, \dots m > 1,$$

$$2. \dot{z} = \pm \frac{rP\dot{y}}{y\sqrt{(ry - P^2)}}, \dots m = 1.$$

$$3. \dot{z} = \pm \frac{r\sqrt{\left(\frac{m}{1-m}\right) \times P\dot{y}}}{y\sqrt{\left(-\frac{m}{1-m} \times P^2 + \frac{r}{1-m} \times y - y^2\right)}}, \dots m < 1.$$

In a conic section if A and B be the transverse and conjugate axes, L the *latus rectum*, y the distance of any point in the section from the focus, and p a perpendicular from thence upon

a tangent to that point; then per conics $p = \frac{\frac{1}{2}By}{\sqrt{(Ay \pm y^2)}}$ the

positive or negative sign taking place according as the curve is an hyperbola or ellipse; but in the parabola, A being infinite,

y^2 will vanish. Hence, because in this case $m = \frac{n-1}{2} \times s^2 = \frac{1}{2} s^2$, it is evident that the first equation belongs to an hyperbola,

A being $= \frac{r}{m-1} = \frac{2r}{s^2-2}$, $B = 2\sqrt{\left(\frac{m}{m-1}\right) \times P} = \frac{2sP}{\sqrt{(s^2-2)}}$

and $L = \frac{B^2}{A} = \frac{4mP^2}{r} = \frac{2s^2 P^2}{r}$. Likewise the perpendicular up-

on the asymptote $= \sqrt{\left(\frac{m}{m-1}\right) \times P} = \frac{sP}{\sqrt{(s^2-2)}}$.

The second is an equation of the parabola; from whence it will appear that $\frac{\frac{1}{2}B}{A^{\frac{1}{2}}} = \frac{1}{2} \sqrt{L} = \frac{P}{r^{\frac{1}{2}}}$; therefore $\frac{1}{2} L = \frac{P^2}{r}$.

The third equation belongs to an ellipse; A being $= \frac{r}{1-m}$
 $= \frac{2r}{2-s^2}$, $B = 2\sqrt{\left(\frac{m}{1-m}\right) \times P} = \frac{2sP}{\sqrt{(2-s^2)}}$ and *latus rectum* $L = \frac{2s^2 P^2}{r}$.

From what has been observed above it will evidently follow, that, if the centripetal force $\propto$ [*vary as*] as the square of the distance from the centre reciprocally, the curve described by the body will be a conic section, or circle.

All

All the common properties of bodies moving in conic sections may easily be deduced from the above equations; but that is not the intention of this proposition.

COR. 3. If $n = 3$, the values of p found in the proposition will become

$$1. p = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times P y}}{\sqrt{\left(y^2 + \frac{r^2}{m-1}\right)}}, \dots\dots\dots m >$$

$$2. p = \frac{P}{r} \times y, \dots\dots\dots m = 1.$$

$$3. p = \frac{\sqrt{\left(\frac{m}{1-m}\right) \times P y}}{\sqrt{\left(\frac{r^2}{1-m} - y^2\right)}}, \dots\dots\dots m < 1.$$

And the respective values of $\dot{z}$ will be

$$1. \dot{z} = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right) \times r P \dot{y}}}{y \sqrt{\left(y^2 + \frac{r^2 - m P^2}{m-1}\right)}} = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right) \times P r \dot{y}}}{y \sqrt{\left(y^2 - \frac{m P^2 - r^2}{m-1}\right)}}, m < 1.$$

$$2. \dot{z} = \pm \frac{P r}{\sqrt{(r^2 - P^2)}} \times \frac{\dot{y}}{y}, \dots\dots\dots m = 1.$$

$$3. \dot{z} = \pm \frac{\sqrt{\left(\frac{m}{1-m}\right) \times r P \dot{y}}}{y \sqrt{\left(\frac{r^2 - m P^2}{1-m} - y^2\right)}}, \dots\dots\dots m > 1.$$

The first value of $\dot{z}$, in which m is greater than unity, may be subdivided into three others, according as $m P^2$ is greater, equal to, or less than r^2 .

1. If $m P^2$ be greater than r^2 , the equation for determining the apfides becomes

$$y^2 - \frac{m}{m-1} P^2 + \frac{r^2}{m-1} = 0; \text{ wherefore } y = \sqrt{\frac{m P^2 - r^2}{m-1}},$$

the curve therefore has an apse and two infinite legs.

By taking the correct fluent it will appear, that

$$z = \frac{r P \sqrt{\left\{ \frac{m(m-1)}{m P^2 - r^2} \right\}}}{m P^2 - r^2} \times \text{the difference of two arcs whose secants}$$

secants are y and r , and radius = $\sqrt{\frac{m P^2 - r^2}{m - 1}} = \frac{\sqrt{(m) \times P}}{\sqrt{(m P^2 - r^2)}}$

$\times$ the difference of two arcs whose secants are $\frac{\sqrt{(m - 1) \times r y}}{\sqrt{(m P^2 - r^2)}}$

and $\frac{\sqrt{(m - 1) \times r^2}}{\sqrt{(m P^2 - r^2)}}$, and radius = r ; hence the trajectory is easily constructed, by finding any number of points at pleasure.

If $P = r$, or the body be projected at right angles to a line drawn from the centre of force, then $z = \sqrt{\left(\frac{m}{m - 1}\right)} \times \text{arch}$

whose secant = y , and radius = r . To construct the orbit upon this supposition; from the centre C (fig. 153, pl. 8.) at the distance $CV = r$, describe the circle VQS ; take any arc VR , and

draw the tangent RT ; take $VQ : VR :: \sqrt{\left(\frac{m}{m - 1}\right)} : 1 ::$

$\sqrt{m} : \sqrt{(m - 1)} :: s : \sqrt{(s^2 - 1)}$, and draw CQ , which produce to n , making $Cn = CT$; then n will be a point in the curve. If VQS be taken to a quadrant in the above proportion, and CS be produced indefinitely, it will be parallel to an asymptote to the curve. This is too evident to require any particular proof. From the centre C draw CP perpendicular to CL , and

take it = $\sqrt{\left(\frac{m}{m - 1}\right)} \times r = \frac{sr}{\sqrt{(s^2 - 1)}}$, and through the point P draw PL parallel to CL , and it will be an asymptote to the curve or trajectory.

The number of revolutions which the body will make, while going from an apse to an infinite distance, will be evidently equal to

$$\frac{\sqrt{\left(\frac{m}{m - 1}\right)} \times \text{quadrant}}{4 \times \text{quadrant}} = \frac{1}{4} \sqrt{\left(\frac{m}{m - 1}\right)}.$$

$$2. \text{ If } m P^2 = r^2, \text{ then } p = \frac{\sqrt{\left(\frac{r^2}{m - 1}\right)} \times y}{\sqrt{\left(y^2 + \frac{r^2}{m - 1}\right)}}, \text{ the equa-}$$

tion of the hyperbolic spiral; the body will therefore revolve in this curve, and will come to an apse at the centre. For at that time

$$y = \sqrt{\frac{m P^2 - r^2}{m - 1}} = 0. \text{ It will likewise appear, that}$$

$\dot{z} = \pm \sqrt{\left(\frac{m}{m-1}\right) \times rP \times \frac{y}{y^2}}$; the correct fluent therefore gives

$$z = \pm \sqrt{\left(\frac{m}{m-1}\right) \times P \times \frac{y-r}{y}} \text{ or } z = \sqrt{\left(\frac{m}{m-1}\right) \times P \times \frac{y-r}{y}},$$

when the body ascends, and $z = \sqrt{\left(\frac{m}{m-1}\right) \times P \times \frac{r-y}{y}}$

when it descends. If the body be infinitely distant from the centre,

then $z = \sqrt{\left(\frac{m}{m-1}\right) \times P}$. Hence if $c = 3.14159$, the

number of revolutions which the body will describe in ascending

to an infinite height from the distance r , $= \sqrt{\left(\frac{m}{m-1}\right) \times P \div 2cr}$

$$= \sqrt{\left(\frac{m}{m-1}\right) \times \frac{P}{2cr}} = \frac{1}{2c \sqrt{\left(\frac{m}{m-1}\right)}} = \frac{1}{2c \sqrt{(j^2-1)}}$$

for $\sqrt{m \times P} = r$. In descending to the centre, it is evident, that

the number of revolutions will be infinite; for when $y = 0$,

$\frac{r-y}{y} = \frac{r}{0}$, is infinite. The distance of the asymptote from

$$\text{the centre} = \sqrt{\left(\frac{m}{m-1}\right) \times P} = \frac{r}{\sqrt{\left(\frac{m}{m-1}\right)}} = \frac{r}{\sqrt{(j^2-1)}}$$

3. If mP^2 be less than r^2 , then the value of y , when the body arrives at an apse, $= \sqrt{\frac{mP^2 - r^2}{m-1}}$, becomes impossible;

therefore upon this supposition the body can never come to an apse. But it may either descend to the centre, or go off to an infinite distance, as is evident from the equation to the curve.

Because $\dot{z} = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right) \times rPj}}{y \sqrt{\left(y^2 + \frac{r^2 - mP^2}{m-1}\right)}}$ by taking the fluent

$$z = \mp rP \sqrt{\left(\frac{m}{r^2 - mP^2}\right) \times \log \frac{\sqrt{\left(y^2 + \frac{r^2 - mP^2}{m-1}\right)} + \sqrt{\left(\frac{r^2 - mP^2}{m-1}\right)}}{y}},$$

and corrected

$$z = \pm rP \sqrt{\left(\frac{m}{r^2 - mP^2}\right) \times \log \frac{\sqrt{\left\{m(r^2 - P^2)\right\}} + \sqrt{(r^2 - mP^2)}}{\sqrt{\left\{(m-1)y^2 + r^2 - mP^2\right\}} + \sqrt{(r^2 - mP^2)}}} \times \frac{y}{r}.$$

If y be infinite, then

$$z = rP \sqrt{\left(\frac{m}{r^2 - mP^2}\right)} \times \log. \frac{\sqrt{\{m(r^2 - P^2)\} + \sqrt{(r^2 - mP^2)}}}{r \sqrt{(m - 1)}}.$$

The body will therefore go off to an infinite distance in a finite number of revolutions.

But if $y = 0$, the value of the fraction

$$\frac{\sqrt{\{m(r^2 - P^2)\} + \sqrt{(r^2 - mP^2)}}}{\sqrt{\{(m-1)y^2 + r^2 - mP^2\} + \sqrt{(r^2 - mP^2)}}} \times \frac{y}{r} \text{ is nothing,}$$

wherefore the body will make an infinite number of revolutions in descending to the centre. From the above equations the trajectory may be constructed.

Next, if $m = 1$, the equation of the trajectory being $p = \frac{P}{r} \times y$, the curve in which the body moves is the log. spiral.

$$\text{But } z = \pm \frac{Pr}{\sqrt{(r^2 - P^2)}} \times \frac{y}{y}, \text{ therefore } z = \frac{Pr}{\sqrt{(r^2 - P^2)}} \times \log. \frac{y}{r}$$

when the body ascends, and $z = \frac{Pr}{\sqrt{(r^2 - P^2)}} \times \log. \frac{r}{y}$ when it

descends, wherefore, the body can neither descend to the centre nor go off to an infinite distance in a finite number of revolutions. If $P = r$, or the body be projected at right angles to the line drawn from the centre, it is evident that it will describe a circle.

In the last case when m is less than unity, the equation of the trajectory is

$$p = \frac{\sqrt{\left(\frac{m}{1-m}\right)} \times Py}{\sqrt{\left(\frac{r^2}{1-m} - y^2\right)}}, \text{ and } z = \pm \frac{\sqrt{\left(\frac{m}{1-m}\right)} \times rPy}{y \sqrt{\left(\frac{r^2 - mP^2}{1-m} - y^2\right)}}.$$

It is evident that when $y = 0$, $p = 0$; the body may there-

fore descend to the centre, but if y be infinite, $p = \frac{\sqrt{\left(\frac{m}{1-m}\right)} \times P}{\sqrt{-1}}$;

therefore the body cannot ascend to an infinite height. It is likewise evident from the above equation that when y increases, p increases; but p can never exceed y , therefore when they are equal, that is, at an apse, y will be the greatest possible; but

then $y = \sqrt{\left(\frac{r^2 - mP^2}{1-m}\right)}$. Wherefore, after the body has

passed an apse, it will descend to the centre,

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Because

Because $\dot{z} = \pm \frac{\sqrt{\left(\frac{m}{1-m}\right) \times r P \dot{y}}}{y \sqrt{\left(\frac{r^2 - m P^2}{1-m} - y^2\right)}}$, the correct flu-

ent being taken, and the equation reduced, we have

$$z = r P \sqrt{\left(\frac{m}{r^2 - m P^2}\right)} \times \begin{cases} + \log. \frac{\sqrt{\left(\frac{r^2 - m P^2}{1-m}\right)} - \sqrt{\left(\frac{r^2 - m P^2}{1-m} - y^2\right)}}{y} \\ - \log. \frac{\sqrt{\left(\frac{r^2 - m P^2}{1-m}\right)} - \sqrt{\left(\frac{r^2 - m P^2}{1-m} - r^2\right)}}{r} \end{cases}$$

If $P = r$, or the body be projected from an apse, then $\dot{z} = r \sqrt{\left(\frac{m}{1-m}\right)} \times \log. \frac{y}{r - \sqrt{(r^2 - y^2)}}$, and when $y = 0$, the value of the fraction $\frac{y}{r - \sqrt{(r^2 - y^2)}}$ is infinite, as may be

found either by taking the fluxions of the numerator and denominator, or by expanding $\sqrt{(r^2 - y^2)}$, and dividing y by the denominator; hence the body will make an infinite number of revolutions before it arrives at the centre.

The orbit may be constructed in the following manner. Let the rectangular hyperbola VA (fig. 154, pl. 8.) be described, whose centre is C and vertex V; join CA, and draw the tangent AT to the point A, and AB perpendicular to the axis VB. Let CV = r , CB = x , and CT = y . It is well known that

the fluxion of the sector CVA = $\frac{r^2}{2} \times \frac{\dot{z}}{\sqrt{(x^2 - r^2)}}$: but $x : r ::$

$r : y$, therefore $x = \frac{r^2}{y}$, and $\dot{x} = -\frac{r^2 \dot{y}}{y^2}$,

consequently $\frac{r^2}{2} \times \frac{\dot{z}}{\sqrt{(x^2 - r^2)}} = -\frac{r^2}{2} \times \frac{r^2 \dot{y}}{y^2 \sqrt{\left(\frac{r^4}{y^2} - r^2\right)}} = -$

$\frac{r^3 \dot{y}}{2y \sqrt{(r^2 - y^2)}}$ fluxion of the sector CVA.

But in this case $\dot{z} = -\frac{\sqrt{\left(\frac{m}{1-m}\right)} \times r^2 \dot{y}}{y \sqrt{(r^2 - y^2)}}$, therefore the

fluxion of the circular sector CVY = $\frac{\sqrt{\left(\frac{m}{1-m}\right)} \times r^3 \dot{y}}{2y \sqrt{(r^2 - y^2)}}$.

Take

Take $\sqrt{\frac{m}{1-m}} : 1 :: \text{hyperbolic sector} : \text{circular sector VCY}$,

and make $Cp = CT$; then p is a point in the trajectory.

From this construction it will easily appear, that the number of revolutions which the body must make before it arrives at the centre will be infinite, because the area CVA increases without limit.

Or the trajectory may be constructed in the following manner. Describe the semicircle SVD, (fig. 155, pl. 8.) with the radius $CV = r$; draw AB perpendicular to CD, and suppose it equal to y , and take the arc VQ = the difference of the hyperbolic logarithms of AB and BD multiplied into $r \sqrt{\frac{m}{1-m}}$, and take $Cp = AB$; then is p a point in the trajectory, as will be evident from the equation $z = r \sqrt{\left(\frac{m}{1-m}\right) \times \log. \frac{y}{r - \sqrt{(r^2 - y^2)}}$.

Because the ratio of AB to BD is infinitely great when AD is evanescent, the number of revolutions before the body arrives at the centre, must be infinite.

$$\text{COR. 4. If } n = 4, \text{ then } p = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times Py^{\frac{3}{2}}}}{\sqrt{\left(y^3 + \frac{r^3}{m-1}\right)}}$$

$$z = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right) \times r Py^{\frac{1}{2}}}}{y \sqrt{\left(y^3 - \frac{m}{m-1} \times P^2 y + \frac{r^3}{m-1}\right)}}, \text{ and } y^3 - \frac{m}{m-1}$$

$P^2 y + \frac{r^3}{m-1} = 0$, is the equation for determining the apsidal;

where m may be greater than, equal to, or less than unity. The ratios of m to unity, and of P to r being given, it will be easy to determine when all the roots of this last equation are real, and when two of them are impossible. If the body be projected from an apse, or $P = r$, then all the roots are real; two being positive and one negative, which belongs to a part of the curve having its concavity turned from the centre of force.

$$\text{If } n = 5, \text{ then } p = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times Py^2}}{\sqrt{\left(y^4 + \frac{r^4}{m-1}\right)}} = \frac{\sqrt{m} \times Py^2}{\sqrt{\{(m-1)y^4 + r^4\}}}$$

$z =$

$$\dot{z} = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times r P y \dot{y}}}{y \sqrt{\left(y^4 - \frac{m}{m-1} P^2 y^2 + \frac{r^4}{m-1}\right)}}$$

and $y^4 - \frac{m}{m-1} P^2 y^2 + \frac{r^4}{m-1} = 0$, is the equation for determining the apsidal; here, likewise, m may be greater than, equal to, or less than unity. The several parts of this curve might easily be traced out when m is greater than unity, but notice will be taken only of that case in which $m = 1$, or the body is projected with the same velocity as it would acquire by falling an infinite height. In this case $p = \frac{P}{r^2} \times y^2$, an equation to a circle, the centre of force being in the circumference, and the diameter $= \frac{r^2}{P}$. For, let C in the circumference of the circle

AVC, (fig. 156, pl. 8.) be the centre of force, CA the diameter, V the point from which the body is projected, n any other place in the circle; draw the tangents VP, np , and the perpendiculars CP, C p , and join V, A and n , A. Then, by similar triangles, CP : CV :: CV : CA, that is, $P : r :: r : CA = \frac{r^2}{P}$; and

$$Cp : Cn :: Cn : CA, \text{ that is, } p : y :: y : \frac{r^2}{P};$$

therefore, $p = \frac{P}{r^2} \times y^2$.

It will easily appear that upon this supposition, the periodic times in different circles would be as the cubes of their diameters directly; taking it for granted that a revolution were possible.

COR. 5. If $n = 1$, or the force be directly as the distance, then, $m = \frac{n-1}{2} \times P^2 = -s^2$; hence, because in general

$$p = \frac{\sqrt{\left(\frac{m}{m-1}\right) \times P y^{\frac{n-1}{2}}}}{\sqrt{\left(y^{n-1} + \frac{r^{n-1}}{m-1}\right)}}, \text{ therefore by substitution}$$

and reduction,

$$p = \frac{s P r}{\sqrt{\{(s^2 + 1)r^2 - y^2\}}}, \text{ and } \dot{z} = \frac{s P r^2 \dot{y}}{y \sqrt{\{-y^4 + (s^2 + 1)r^2 y^2 - s^2 P^2 r^2\}}}$$

The

The above is an equation to an ellipse, the centre of force being in its centre. For, if $2R$ = the transverse, and $2C$ = the conjugate axis, y = the distance from the centre, and p =

the perpendicular upon the tangent, then $p = \frac{CR}{\sqrt{\{(R^2 + C^2) - y^2\}}}$.

Compare this with the above equation, and it will easily appear, that

$$2R = \sqrt{\{(s^2 + 1)r^2 + 2sPr\}} + \sqrt{\{(s^2 + 1)r^2 - 2sPr\}}, \text{ and}$$

$$2C = \sqrt{\{(s^2 + 1)r^2 + 2sPr\}} - \sqrt{\{(s^2 + 1)r^2 - 2sPr\}}.$$

The same conclusion may likewise be deduced from the equation for determining the apsides, which has two roots positive, and two equal to them and negative.

COR. 6. If $m = 1$, then $s^2 = \frac{2}{n-1}$; which, if n be greater

than 3, is less than unity, and the body in this case must fall to the centre; and the number of revolutions it will make before it arrives there, may be determined in the following manner. In

this case $p = \frac{P}{\frac{n-1}{2}} \times y^{\frac{n-1}{2}} = \frac{y^{\frac{n-1}{2}}}{r^{\frac{n-3}{2}}}$ when $P = r$, or the body

moves from an apse; therefore $p : y :: y^{\frac{n-3}{2}} : r^{\frac{n-3}{2}}$. But when

y and p are evanescent, $r^{\frac{n-3}{2}}$ is infinitely greater than $y^{\frac{n-3}{2}}$, therefore y is infinitely greater than p , and consequently at that time the angle nCp (fig. 152, pl. 8.) will be a right angle. From the

equation of the curve, it will easily appear that $\frac{p}{y} = \frac{n-1}{2} \times \frac{\dot{y}}{y}$. Let

A and a represent the angles described by Cp , and Cn , respectively,

since the body left an apse, then because $\dot{A} : \dot{a} :: \frac{p}{y} : \frac{\dot{y}}{y} ::$
 $n - 1$

* If a body revolve in a curve of any kind round a centre of force; to compare the angular velocity of the perpendicular upon the tangent with that of the distance from the centre, or radius vector.

Let PQW (fig. 157, pl. 8.) be the curve in which the body moves, S the centre of force, and C the centre of curvature. Let P, Q, be two points in the curve indefinitely near to each other, to which the tangents PY, QY are drawn; let

$\frac{n-1}{2} : 1, \dot{A} = \frac{n-1}{2} \dot{a}$, and $A = \frac{n-1}{2} a$; for A and a begin together. But $A = Q + a$ when $y = 0$, or the body arrives at the centre; hence we have $a = \frac{2}{n-3} \times Q = \frac{1}{2(n-3)} \times C$, Q being a quadrant, and $C =$ the circumference of a circle whose radius $= r$. The number of revolutions therefore $= \frac{1}{2(n-3)}$.

If n be less than 3, s must be greater than unity; therefore the body projected at right angles to a line drawn from the centre of force, and with a greater velocity than would make it describe a circle, it must begin to ascend; and it must ever continue rising as its velocity is equal to that acquired by falling from an infinite

height. Because $p : y :: r^{\frac{3-n}{2}} : y^{\frac{3-n}{2}}$, therefore when y is infinite, p is finite, and hence the angle nCp , at that time is a right angle. As above $A = \frac{n-1}{2} \times a$, and $a = Q + A$,

therefore $a = \frac{2}{3-n} \times Q = \frac{1}{2(3-n)} \times C$. Hence the number of revolutions the body describes in ascending to an infinite height $= \frac{1}{2(3-n)}$.

COR. 7. If the centre repel the body, or the force be a centrifugal one, then $\frac{P^2 v^2 \dot{p}}{p^3 \dot{y}} = - \frac{A^{n+1}}{y^n}$;

multiply both sides by $\dot{y}$ and take the correct fluents,

$$\text{then } \frac{P^2 v^2}{2p^2} - \frac{v^2}{2} = \frac{A^{n+1}}{(n-1)r^{n-1}} \times \frac{y^{n-1} - r^{n-1}}{y^{n-1}}.$$

But

let fall the perpendiculars SY, Sy , and QT , which last may be taken for the arch of a circle described from the centre S . It is evident that the angles PSQ, PCQ , are to each other

$$= \frac{QT}{SP} : \frac{PQ}{CP} :: \frac{SY}{SP} : \frac{SP}{CP} \text{ (by similar triangles,) } :: \frac{p}{y} : \frac{y}{CP}.$$

But $CP = \frac{y^2}{p}$, and the angle $PCQ = YSy$,

$$\text{therefore the angle } PSQ : YSy :: \frac{p}{y} : y \div \frac{y^2}{p} : \frac{y}{y} : \frac{p}{p}.$$

But $\frac{2mA^{n+1}}{(n-1)r^{n-1}} = v^2$, hence, by reduction,

$$p^2 = \frac{mP^2 y^{n-1}}{(m+1)y^{n-1} - r^{n-1}}, \text{ and because } m = \frac{n-1}{2} s^2, \text{ therefore}$$

$$p^2 = \frac{(n-1)s^2 P^2 y^{n-1}}{\{(n-1)s^2 + 2\} \cdot y^{n-1} - 2r^{n-1}} \text{ and } p = \frac{\sqrt{(n-1) \times sP y^{\frac{n-1}{2}}}}{\sqrt{\{(n-1)s^2 + 2\} y^{\frac{n-1}{2}} - 2r^{\frac{n-1}{2}}}}$$

the equation of the curve described.

By substitution it will likewise appear, that

$$\dot{z} = \frac{\sqrt{(n-1) \times srP y^{\frac{n-3}{2}}} \dot{y}}{y \sqrt{\{(n-1)s^2 + 2\} \cdot y^{n-1} - (n-1)s^2 P^2 y^{n-3} - 2r^{n-1}}}$$

And for determining the apsidal we have this equation, viz.

$$y^{n-1} - \frac{(n-1)s^2}{(n-1)s^2 + 2} \times P^2 y^{n-3} - \frac{2}{(n-1)s^2 + 2} \times r^{n-1} = 0.$$

If $n = 1$, then by proceeding in the very same manner as above, we shall find

$$p = \frac{P}{\sqrt{(1 + \frac{2}{s^2} \times \text{h.l. } \frac{y}{r})}} \text{ and } \dot{z} = \frac{rP\dot{y}}{y \sqrt{\{(1 + \frac{2}{s^2} \text{h.l. } \frac{y}{r})y^2 - P^2\}}}$$

$$\text{If } n = 2, \text{ then } p = \frac{sPy}{\sqrt{\{(s+2)y^2 - 2ry\}}} = \frac{sP}{\sqrt{(s+2)}} \times \frac{y}{\sqrt{(y^2 - \frac{2r}{s+2} \times y)}}$$

an equation to the opposite hyperbola, the transverse axis

$$\text{being} = \frac{2r}{s^2 + 2} \text{ and conjugate} = \frac{2sP}{\sqrt{(s^2 + 2)}}.$$

If $n = -1$, then $p = \frac{sPr}{\sqrt{\{y^2 + (s^2 - 1)r^2\}}}$, an equation to an hyperbola, the centre of force being in the centre of the hyperbola; and if $s = 1$, then $p = \frac{Pr}{y}$, the trajectory is therefore the rectangular hyperbola.

In the two last cases the value of z may be found by circular arcs and logarithms.

SCHOLIUM.

Because $\frac{v}{s}$ = velocity in a circle at the distance r from the centre, and if the force $\propto \frac{1}{y^n}$, the velocities in circles will $\propto \frac{1}{y^{\frac{n-1}{2}}}$

(Principia. prop. 4.) consequently $\frac{r^{\frac{n-1}{2}}}{y^{\frac{n-1}{2}}} \times \frac{v}{s}$ = velocity in a cir-

cle at the distance y from the centre. But $\frac{Pv}{p}$ = velocity in the trajectory at the distance y ; if, therefore, we make

$\frac{Pv}{p} = \frac{r^{\frac{n-1}{2}}}{y^{\frac{n-1}{2}}} \times \frac{v}{s}$, and at the same time the body be supposed to

arrive at an apse, in which case $p^2 = y^2 = \frac{\frac{m}{m-1} \times P^2 y^{n-1}}{y^{n-1} + \frac{r^{n-1}}{m-1}}$,

it would continue to move for ever in this circle. But coming to an apse, it must ascend or descend in a similar and equal curve, hence it never can arrive at the distance y from the centre, determined from the above equations, in any finite number of revolutions.

Making, therefore, as above $\frac{Pv}{p} = \frac{r^{\frac{n-1}{2}}}{y^{\frac{n-1}{2}}} \times \frac{v}{s}$, we have

$$p^2 = P^2 s^2 \times \frac{y^{n-1}}{r^{n-1}} = \frac{\frac{m}{m-1} \times P^2 y^{n-1}}{y^{n-1} + \frac{r^{n-1}}{m-1}},$$

$$\text{hence, } s^2 = \frac{\frac{m}{m-1} \times r^{n-1}}{y^{n-1} + \frac{r^{n-1}}{m-1}}.$$

But

But because $p^2 = y^2 = \frac{\frac{m}{m-1} \times P^2 y^{n-1}}{y^{n-1} + \frac{r^{n-1}}{m-1}},$

$$y^{n-1} + \frac{r^{n-1}}{m-1} = \frac{m}{m-1} \times P^2 y^{n-3}; \quad s^2 P^2 = \frac{r^{n-3}}{y^{n-3}}.$$

But the equation for determining the apfides is

$$y^{n-1} - \frac{m}{m-1} P^2 y^{n-3} + \frac{r^{n-1}}{m-1} = 0, \text{ or } ,$$

$$y^{n-1} - \frac{n-1}{(n-1)s^2-2} \times s^2 P^2 y^{n-1} + \frac{2r^{n-3}}{(n-1)s^2-2} = 0;$$

whence by substitution we have

$$y^{n-1} - \frac{(n-1)r^{n-1}}{(n-1)s^2-2} + \frac{2r^{n-1}}{(n-1)s^2-2} = 0, \text{ or } y^{n-1} = \frac{n-3}{(n-1)s^2-2} \times r^{n-1},$$

$$\text{therefore } y = \left(\frac{n-3}{(n-1)s^2-2} \right)^{\frac{1}{n-1}} \times r = \left(\frac{n-3}{2m-2} \right)^{\frac{1}{n-1}} \times r,$$

because $\frac{2m}{n-1} = s^2$. It is evident from inspection, that n

must be greater than 3 and m greater than unity; the force therefore must vary in an higher proportion than the cube of the distance inversely. If the body descend from the point V,

then $s^2 = \frac{r^{n-1}}{P^2 y^{n-3}}$, must be greater than unity. If it ascend,

then $n-3$ must be greater than $(n-1)s^2-2$, and therefore s^2 , and consequently s , less than unity.

From the above equations and observations, we have the following construction. Fig. 158, pl. 8. Let C be the centre of force, V the point from which the body is projected in the direction VP, which makes an acute angle with CV, and with a velocity greater than would make it move in the circle VZU if projected at right angles. With the centre C and radius

$Ca = \left(\frac{n-1}{(n-1)s^2-2} \right)^{\frac{1}{n-1}} \times r$ describe the circle *abd*, and

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from

from C upon VP let fall the perpendicular CP, making

it equal to $\frac{r^{\frac{n-1}{2}}}{s \times (Ca)^{\frac{n-1}{2}}}$. Then, from what has been deter-

mined in this Scholium, it will be evident, that the body will move in the curve VW, to which VP is a tangent, continually descending towards the circle *abd*, but will never arrive at it in any finite number of revolutions. This circle is therefore an asymptote to the trajectory.

In the same manner if *s* be less than unity, or the velocity with which the body is projected in a line VT, which makes an acute angle with VA, be less than the velocity of a body in the circle VUZ; then with the centre C and radius

$CA = \left(\frac{n-3}{(n-1)s^2-2} \right)^{\frac{1}{n-1}} \times r$ describe the circle ABD, and

from C upon TV produced let fall the perpendicular CP,

which make equal to $\frac{r^{\frac{n-1}{2}}}{s \times (CA)^{\frac{n-1}{2}}}$; then the body will

continually ascend from the centre, but will never arrive at the circle ABD. This circle, therefore, is likewise an asymptote to the curve in which the body moves.

Hence, if a body be projected from any point, and descend towards the centre, the velocity with which it is projected must be greater than either that which it would acquire in falling from an infinite height to that point, or than that of a body describing the circle at the distance of that place from the centre of force. If it ascend, the velocity must be greater than that acquired by descending from an infinite height, but less than that of a body in a circle at that distance. Within these limits the construction is general.

PROPOSITION II.

The same things being given as in the last proposition, to determine the velocity and time corresponding to any given distance of the body from the centre of force.

Let

Let the velocities of bodies in a circle at the distance r in the curve, at V and n , be V, v, v' respectively (fig. 152, pl. 8.); then $v' = \frac{Pv}{p}$. If therefore the several values of p , depending upon the different values of m , be substituted, we shall have

$$1. v' = \frac{v \sqrt{\left(y^{\frac{n-1}{2}} + \frac{r^{\frac{n-1}{2}}}{m-1} \right)}}{\sqrt{\left(\frac{m}{m-1} \right) \times y^{\frac{n-1}{2}}}} = v \sqrt{\frac{(m-1)y^{\frac{n-1}{2}} + r^{\frac{n-1}{2}}}{my^{\frac{n-1}{2}}}}, \quad m > 1.$$

$$2. v' = v \sqrt{\frac{r^{\frac{n-1}{2}}}{y^{\frac{n-1}{2}}}}, \quad \dots \dots \dots m = 1.$$

$$3. v' = \frac{v \sqrt{\left(\frac{r^{\frac{n-1}{2}}}{1-m} - y^{\frac{n-1}{2}} \right)}}{\sqrt{\left(\frac{m}{1-m} \right) \times y^{\frac{n-1}{2}}}} = v \sqrt{\frac{r^{\frac{n-1}{2}} - (1-m)y^{\frac{n-1}{2}}}{my^{\frac{n-1}{2}}}}, \quad m < 1.$$

$$4. v' = \frac{v \sqrt{\left(\frac{r^{\frac{n-2}{2}}}{2} - y^{\frac{n-1}{2}} \right)}}{\sqrt{\left(\frac{n-1}{2} \right) \times y^{\frac{n-1}{2}}}}, \quad m = 0, \text{ therefore } v = 0.$$

If $n=1$, then $v' = v \sqrt{\left(1 + \frac{2}{s^2} \times \log. \frac{r}{y} \right)}$. In the 4th value of $v', m=0 = s^2 \times \frac{n-1}{2}$, $v : V :: s : 1$; therefore $sV = v$; hence

$$\frac{m}{1-m} = \frac{s^2 (n-1)}{2 - s^2 (n-1)}, \text{ and } \frac{v}{\sqrt{\frac{m}{1-m}}} = \frac{V}{\sqrt{\frac{n-1}{2}}}, \text{ } s \text{ being } = 0,$$

therefore the 4th form is evident from the 3d. Because $m = s^2 \times \frac{n-1}{2}$, and $s^2 = \frac{v^2}{V^2}$, therefore $m = \frac{v^2}{V^2} \times \frac{n-1}{2}$; which value of m being substituted for it in the first equation gives, when reduced

$$v' = \sqrt{\left(v^2 - 2V^2 \times \frac{y^{\frac{n-1}{2}} - r^{\frac{n-1}{2}}}{(n-1)y^{\frac{n-1}{2}}} \right)} = \sqrt{\left(v^2 + 2V^2 \times \frac{r^{\frac{n-1}{2}} - y^{\frac{n-1}{2}}}{(n-1)y^{\frac{n-1}{2}}} \right)}.$$

Let g = force of gravity, and suppose the force at V , or at the distance r from the centre of force : force of gravity (g) :: $l : 1$,

then

then $lg =$ force at P , measured by the velocity in the time (1);

hence, $\frac{V^2}{r} = lg$, $V^2 = lgr$, therefore by substitution

$$v^2 = \sqrt{\left(v^2 - 2lgr \times \frac{y^{n-1} - r^{n-1}}{(n-1)y^{n-1}} \right)} = \sqrt{\left(v^2 + 2lgr \times \frac{r^{n-1} - y^{n-1}}{(n-1)y^{n-1}} \right)}.$$

When $n = 1$, if the above value of s^2 be substituted we have

$$v' = \sqrt{\left(v^2 + 2lgr \times \log. \frac{r}{y} \right)} = \sqrt{\left(v^2 - 2lgr \times \log. \frac{y}{r} \right)}.$$

Because P does not enter into the values of v' , it is evident that the velocity of the body will be the same, whether it move in a curve line, or directly to or from the centre; the distance of y being the same in both cases.

By the help of the 4th theorem, it will be easy to determine how far a body must fall towards the centre, to acquire the velocity it has in the curve. For the value of v' there given being put $= v = sV$, we shall have by reduction

$$y = r \times \left(\frac{2}{s^2(n-1) + 2} \right)^{\frac{1}{n-1}}$$

and the space descended

$$r - y = r \times \frac{\left(\frac{2}{s^2(n-1) + 2} \right)^{\frac{1}{n-1}} - \left(\frac{2}{2} \right)^{\frac{1}{n-1}}}{\left(\frac{2}{s^2(n-1) + 2} \right)^{\frac{1}{n-1}}}.$$

If $n = 2$, then $r - y = \frac{s^2}{s^2 + 2} \times r$, which is general for all the conic sections. In the hyperbola, the transverse axis $= \frac{2r}{s^2 - 2} =$

suppose, to A (see cor. 2. prop. 1.), therefore $s^2 = \frac{2A + 2r}{A}$;

hence by substitution $r - y = \frac{A + r}{2A + r} \times r$.

In the parabola, $r - y = \frac{s^2}{s^2 + 2} = \frac{1}{2}r$, s^2 being in this case $= 2$.

In the ellipsis, $r - y = \frac{A - r}{2A - r} \times r$, the transverse axis being

$$\frac{2r}{2 - s^2} = A.$$

If the body be supposed to move in a circle, and therefore $s = 1$, then

$$r - y = \frac{(n+1)^{\frac{1}{n-1}} - (2)^{\frac{1}{n-1}}}{(n+1)^{\frac{1}{n-1}}} \times r = \left\{ 1 - \left(\frac{2}{n+1} \right)^{\frac{1}{n-1}} \right\} \times r.$$

If $n = 2$, then $r - y = \frac{1}{2} r$.

If it be required to find how high a body will ascend, if projected upwards with the velocity it has in the curve; or how far it must fall externally, to acquire that velocity, put the value of v' , in the third theorem, $= 0$, then

$$y^{n-1} = \frac{r^{n-1}}{1-m} = \frac{2r^{n-1}}{2-s^2(n-1)}, \text{ and } y = r \times \left(\frac{2}{2-s^2(n-1)} \right)^{\frac{1}{n-1}}.$$

If $n = 2$, then $y = r \times \frac{2}{2-s^2} = A$, the greater axis of the ellipse.

If $s = 1$, then $y = r \times \frac{2}{2-s^2} = 2r$.

Lastly, when m is greater than unity, it may be determined by the first theorem, what proportion the velocity in the curve bears to that velocity, towards which it continually approaches as the body recedes indefinitely from the centre. For, if y be supposed infinite, then

$$v' \text{ will become } = \frac{v}{\sqrt{\frac{m}{m-1}}} = \frac{v}{\sqrt{\frac{s^2(n-1)}{s^2(n-1)-2}}}. \text{ Hence,}$$

the velocity in the curve : to velocity at an infinite distance

$$:: v : \frac{v}{\sqrt{\frac{s^2(n-1)}{s^2(n-1)-2}}} :: \sqrt{\frac{s^2(n-1)}{s^2(n-1)-2}} : 1.$$

In the hyperbola, that is, when $n = 2$, the proportion will be

$$\sqrt{\frac{s^2}{s^2-2}} : 1 :: \sqrt{(A+r)} : \sqrt{r}; A \text{ being equal the transverse axis.}$$

Let $t =$ time of moving from V to n , $x =$ curve line Vn (fig.

$$152. \text{ pl. 8.}); \text{ then } \sqrt{(y^2 - p^2)} : y :: \dot{y} : \dot{x} = \frac{y\dot{y}}{\sqrt{(y^2 - p^2)}}.$$

$$\text{But } v' : \dot{x} :: 1 : \dot{t} = \frac{\dot{x}}{\dot{t}} = \frac{p\dot{x}}{p\dot{t}}, \text{ because } v' = \frac{pv}{p},$$

therefore

therefore $\bar{i} = \pm \frac{py\dot{y}}{Pv\sqrt{(y^2 - p^2)}}$, in which expression, if the several values of p be substituted, we shall have

$$1. \bar{i} = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right) \times y^{\frac{n-1}{2}} \dot{y}}}{v \sqrt{\left(y^{n-1} - \frac{m}{m-1} \times P^2 y^{n-3} + \frac{r}{m-1}\right)}}, \quad m \text{ being greater than } 1.$$

$$2. \bar{i} = \pm \frac{y^{\frac{n-1}{2}} \dot{y}}{v \sqrt{(r^{n-1} - P^2 y^{n-3})}}, \quad \dots \dots \dots m = 1.$$

$$3. \bar{i} = \pm \frac{\sqrt{\left(\frac{m}{1-m}\right) \times y^{\frac{n-1}{2}} \dot{y}}}{v \sqrt{\left\{\left(\frac{r^{n-1}}{1-m}\right) - \frac{m}{1-m} \times P^2 y^{n-3} - y^{n-1}\right\}}}, \quad m \text{ being less than } 1.$$

$$4. \bar{i} = \pm \frac{\sqrt{\left(\frac{n-1}{2}\right) \times y^{\frac{n-1}{2}} \dot{y}}}{V \sqrt{(r^{n-1} - y^{n-1})}}, \quad P = 0, v = 0 \text{ and } m = 0.$$

The 4th theorem is found by making the same substitution as before.

COR. 1. If $n = 2$, the several values of $\bar{i}$ will become

$$1. \bar{i} = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right) \times y\dot{y}}}{v \sqrt{\left(y^2 + \frac{r}{m-1} \times y - \frac{m}{m-1} \times P^2\right)}}, \quad m \text{ being greater than } 1.$$

$$2. \bar{i} = \pm \frac{y\dot{y}}{v \sqrt{(ry - P^2)}}, \quad \dots \dots \dots m = 1.$$

$$3. \bar{i} = \pm \frac{\sqrt{\left(\frac{m}{1-m}\right) \times y\dot{y}}}{v \sqrt{\left(-\frac{m}{1-m} \times P^2 + \frac{r}{1-m} \times y - y^2\right)}}, \quad m \text{ being less than } 1.$$

$$4. \bar{i} =$$

$$4. i = \pm \frac{\sqrt{\frac{1}{2}} \times y\dot{y}}{V \sqrt{(ry - y^2)}} = \frac{\sqrt{\frac{1}{2}} \times y\dot{y}}{\sqrt{(lgr)} \times \sqrt{(ry - y^2)}}, P=0, m=0.$$

The fluents in all these cases may be found by circular arcs and logarithms.

The correct fluent being taken for the 4th. form, gives

$$i = \frac{1}{V \sqrt{\frac{1}{2}}} \times \left\{ \begin{array}{l} \sqrt{(ry - y^2)} + \text{arc whose versed sine} = r - y \\ \text{and radius} = \frac{1}{2} r \end{array} \right\}$$

$$= \frac{1}{\sqrt{(2lgr)}} \times \left\{ \begin{array}{l} \sqrt{(ry - y^2)} + \text{arc whose versed sine} = r - y \\ \text{and radius} = \frac{1}{2} r \end{array} \right\}$$

and when the body is fallen to the centre, then

$$t = \frac{1}{\sqrt{(2lgr)}} \times \frac{pr}{2}; p \text{ being} = 3.141592.$$

COR. 2. If $n = 3$ the values of i will become

$$1. i = \pm \frac{\sqrt{\left(\frac{m}{m-1}\right)}}{v \sqrt{\left(y^2 + \frac{r^2 - mP^2}{m-1}\right)}}, \dots\dots\dots m > 1.$$

$$2. i = \pm \frac{y\dot{y}}{v \sqrt{(r^2 - P^2)}}, \dots\dots\dots m = 1.$$

$$3. i = \pm \frac{y\dot{y}}{v \sqrt{\left(\frac{r^2 - mP^2}{1-m} - y^2\right)}}, \dots\dots\dots m < 1.$$

$$4. t = - \frac{y\dot{y}}{V \sqrt{(r^2 - y^2)}} = - \frac{y\dot{y}}{\sqrt{(lgr)} \times \sqrt{(r^2 - y^2)}}, P=0, m=0.$$

Hence by taking the correct fluents there will arise

$$1. t = \frac{1}{v} \sqrt{\left(\frac{m}{m-1}\right)} \times \left\{ \pm \sqrt{\left(y^2 + \frac{r^2 - mP^2}{m-1}\right)} \mp \sqrt{\left(\frac{m}{m-1}\right)} \times (r^2 - P^2) \right\}, m > 1.$$

$$2. t = \frac{1}{2v} \times \frac{y^2 \mp r^2}{\sqrt{(r^2 - P^2)}}, \dots\dots\dots m = 1.$$

$$3. t = \frac{1}{v} \sqrt{\left(\frac{m}{1-m}\right)} \times \left\{ \mp \sqrt{\left(\frac{r^2 - mP^2}{1-m} - y^2\right)} \pm \sqrt{\left[\left(\frac{m}{1-m}\right) \times (r^2 - P^2)\right]} \right\}, m < 1.$$

$$4. t = \frac{1}{V} \times \sqrt{(r^2 - y^2)} = \frac{r}{V} = \frac{r}{\sqrt{(lgr)}} = \frac{r^{\frac{1}{2}}}{\sqrt{(lg)}}, \text{ when } y=0, m=0.$$

Consequently, at the end of any given time, the place of the body may be found. For, from the equations given above, the value of y may be found if t be known, and y being known, z , or the arc VY will be given.

If the time of moving over any given space in a right line directly from, or towards the centre be required, it will immediately be found by making $P = 0$, in the three first of the above theorems.

PROPOSITION III.

If a body be acted upon by two forces tending to the same centre, which vary as the n th and q th powers of the distance reciprocally; it is required to determine the equation of the orbit it will describe, &c.

Suppose the whole force acting upon the body, at the distance

$$y \text{ from the centre to be } = \frac{A^{n+1}}{y^n} + \frac{B^{q+1}}{y^q};$$

hence by proceeding as in the first proposition, we shall have

$$\frac{P^2 v^2}{2p^2} - \frac{v^2}{2} = \frac{A^{n+1}}{(n-1)r^{n-1}} \times \frac{r^{n-1} - y^{n-1}}{y^{n-1}} + \frac{B^{q+1}}{(q-1)r^{q-1}} \times \frac{r^{q-1} - y^{q-1}}{y^{q-1}}.$$

$$\text{Let } \frac{2m'B^{q+1}}{(q-1)r^{q-1}} = v^2, \text{ then there arises}$$

$$\frac{P^2}{p^2} - 1 = \frac{r^{n-1} - y^{n-1}}{my^{n-1}} + \frac{r^{q-1} - y^{q-1}}{m'y^{q-1}}. \text{ Hence by}$$

reduction it will appear that

$$p^2 = \frac{mm'P^2y^{n-1}}{(mm' - m - m')y^{n-1} + mr^{q-1}y^{n-q} + m'r^{n-1}}$$

$$= \frac{mm'P^2y^{q-1}}{(mm' - m - m')y^{q-1} + mr^{n-1}y^{q-n} + m'r^{q-1}}.$$

$$\text{Suppose } \frac{B^{q+1}}{r^q} = \frac{cA^{n+1}}{r^n}, \text{ then since } \frac{2mA^{n+1}}{(n-1)r^{n-1}} = (v^2 =)$$

$$\frac{2m'B^{q+1}}{(q-1)r^{q-1}};$$

$$\frac{mA^{n+1}}{n-1} = \frac{cm'A^{n+1} \times r^{q-n}}{(q-1)r^{q-n}}, \text{ or, } \frac{m}{n-1} = \frac{cm'}{q-1},$$

hence $m' = \frac{(q-1)m}{c(n-1)}.$

If therefore this value of m' be substituted in the above value of p^2 , there will arise for the general equation of the curve

$$p^2 = \frac{mP^2y^{n-1}}{r^{n-1} + c \times \frac{n-1}{q-1} \times r^{q-1}y^{n-q} + (m-1-c) \times \frac{n-1}{q-1} \times y^{n-1}}.$$

If p be supposed $= y$, then the equation for determining the distance of the apsidal from the centre of force will become

$$y^{n-1} + \frac{c(n-1)r^{q-1}}{(m-1)(q-1)-c(n-1)} \times y^{n-q} - \frac{m(q-1)P^2}{(m-1)(q-1)-c(n-1)} \left. \begin{array}{l} \times y^{n-3} + \frac{(q-1)r^{n-1}}{(m-1)(q-1)-c(n-1)} \end{array} \right\} = 0.$$

The value of z will be found by an easy substitution.

After the same manner the equation of the curve, &c. &c. may be found, if three or more forces act upon a body towards the same centre.

COR. If $n=2$, $q=-1$, $r=P$, and the force $\frac{B^{q+1}}{y^q}$, be sup-

posed to act from the centre, or, which is the same thing, $-c$ be put for $+c$; then the equation for determining the distance of the apsidal from the centre will become

$$y^4 - \frac{2+c-2m}{c} \times r^2y^2 + \frac{2r^3}{c} \times y - \frac{2mr^4}{c} = 0. \text{ It is evident}$$

that one of the values of y will be $= r$, and the equation given above being divided by $y-r$, the quotient will be

$$y^3 + ry^2 - \frac{2-2m}{c} \times r^2y + \frac{2m}{c} \times r^3 = 0; \text{ by solving which}$$

equation, the distance of the other apse from the centre may be determined.

If it be required to exterminate m , we have

$$\frac{v^2}{r^2} \times \frac{1}{r} = \frac{A^{n+1}}{r^{n-1}} + \frac{B^{q+1}}{r^q} \text{ (Proposition I.) or}$$

$$m^2 = r^2 \times \left(\frac{A^{n+1}}{r^{n-1}} + \frac{B^{q+1}}{r^q} \right) = \frac{2mA^{n+1}}{(n-1)r^{n-1}};$$

(i 2)

and

and by this proposition, $B^{q+1} = \frac{cA^{n+1}}{r^{n-q}}$, therefore

$$s^2(1+c) = \frac{2m}{n-1}, \text{ or } m = \frac{s^2(n-1)(1+c)}{2} = \frac{s^2(1-c)}{2} \text{ in this}$$

corollary; wherefore by substituting this value in the above equation, we have

$$y^3 + ry^3 - \frac{2-s^2(1-c)}{c} \times r^2y + \frac{s^2(1-c)}{c} \times r^3 = 0. \text{ But}$$

$$\begin{aligned} \dot{x} &= \frac{r\dot{p}\dot{y}}{y\sqrt{(y^2-p^2)}} = \frac{\sqrt{(2m)} \times r^3\dot{y}}{y\sqrt{\{cy^4 - (2+c-2m)r^2y^2 + 2r^3y - 2mr^4\}}} \\ &= \frac{s\sqrt{(1-c)} \times r^3\dot{y}}{y\sqrt{\{cy^4 - [2+c-s^2(1-c)] \times r^2y^2 + 2r^3y - s^2(1-c)r^4\}}} \end{aligned}$$

if the fluent therefore be taken, when y = the distance of the other apse from the centre, the arc described in passing from one apse to another; and, consequently, the motion of the apses will be found, whatever be the form of the orbit. Hence it is evident, that if s and c be given, the eccentricity of the orbit and the motion of the apses may be calculated.

SCHOLIUM.

If the force which varies as the q power of the distance reciprocally, act from the centre; or, which is the same thing, c be negative, then

$$\dot{p}^2 = \frac{mP^2y^{n-2}}{r^{n-1} - c \times \frac{n-1}{q-1} \times r^{q-1}y^{n-q} + (m-1+c) \times \frac{n-1}{q-1} \times y^{n-1}}$$

At the distance y from the centre, the two forces become

$\frac{r^n}{y^n}$, and $\frac{cr^q}{y^q}$ which being made equal to each other, the value of y will be determined, where the curve will have a point of con-

trary flexure, viz. $y = \frac{1}{\frac{1}{c^{q-n}}} \times r = \frac{1}{c^{q-n}} \times r$; which will be

greater or less than r , according as n is greater or less than q ; c being less than unity.

Substitute

Substitute this value of y in the above equation, and at the same time suppose $y = p$ and n greater than q ; then the point of contrary flexure will be at an apse: therefore a similar and equal curve will be described on the other side the apse that was described before: but this is impossible, as the curve is now convex towards the centre of force; wherefore the body, upon this supposition, can never arrive at an apse, but will continually approach nearer and nearer to a circle described at the distance y determined above; which will therefore be an asymptote to the orbit or spiral described. By substitution

$$\frac{2}{c^{q-n}} \cdot r^2 = \frac{m P^2 c^{q-n} \cdot r^{n-1}}{r^{n-1} \cdot c \cdot \frac{n-1}{q-1} \cdot r^{q-1} \cdot c^{\frac{n-q}{q-n}} \cdot r^{n-q} + \left(m-1+c, \frac{n-1}{q-1} \right) \cdot c^{\frac{n-1}{q-n}} + r^{n-1}}$$

$$= \frac{m P^2 c^{\frac{n-1}{q-n}}}{\left(m-1+c \times \frac{n-1}{q-1} \right) \times c^{\frac{n-1}{q-n}} - \frac{n-q}{q-1}}$$

$$\text{or } r^2 = \frac{m P^2 c^{\frac{n-3}{q-n}}}{\left(m-1+c \times \frac{n-1}{q-1} \right) \times c^{\frac{n-1}{q-n}} - \frac{n-q}{q-1}}$$

$$\text{But } m = \frac{s^2 (n-1) (1-c)}{2}$$

whence, by substitution and reduction,

$$s^2 = \frac{2r^2 \times (c^{\frac{n-1}{q-n}} \times \left\{ 1 - c \times \frac{n-1}{q-1} \right\} + \frac{n-q}{q-1})}{(n-1)(1-c) \times \left(r^2 c^{\frac{n-1}{q-n}} - P^2 c^{\frac{n-3}{q-n}} \right)}$$

Hence the velocity is determined with which a body must be projected, so as never to arrive at an apse, nor ascend beyond, or descend to a circle, described at the distance $c^{\frac{1}{q-n}} \times r$ from the centre of force.

If $n = 2$, $q = -1$, and $P = r$, then

$$s^2 = \frac{2 + c - 3c^{\frac{1}{3}}}{(1-c)(1-c^{\frac{1}{3}})} = \frac{2 + c - 3c^{\frac{1}{3}}}{(1-c)(1+c^{\frac{1}{3}})(1-c^{\frac{1}{3}})}$$

Wherefore,

Wherefore, if the velocity were known with which a body (as the moon) would move in a circle round another (as the earth) at the distance r of the lower apse, with the compound force $1 - c$; then the velocity is determined with which it must be projected from that apse, to move in the manner described in this scholium; or in such manner as never to arrive at another in any finite number of revolutions.

Hence it will easily appear, that a *very small* variation in the value of s , will occasion a very large one in the excentricity of the orbit, and in the motion of the apsides: it is therefore evident that the two last will increase or decrease at the same time.

PROPOSITION IV.

The centripetal force being reciprocally as the n th power of the distance from a plane parallel to the horizon, and the direction and velocity of a body at any point being given, it is required to determine the nature of the curve it will describe.

Let r = distance of the point at which the body is projected from the horizontal plane, b = velocity parallel to, and c = velocity perpendicular to the plane, x = any abscissa, y = the corresponding ordinate, and z the curve described; then

$\dot{x} : \dot{z} :: b : \frac{b\dot{z}}{x}$ = velocity of the body in the curve.

The force in direction of the ordinate is = the square of the velocity divided by $\frac{1}{2}$ chord of curvature perpendicular to the horizon, or passing through the centre of force. But $\frac{1}{2}$ chord, of curvature when x is constant, which is the case at present

= $-\frac{\dot{z}^2}{y}$; therefore the centripetal force

$$= \frac{b^2 \dot{z}^2}{x^2} \div -\frac{\dot{z}^2}{y} = -\frac{b^2 \ddot{y}}{x^2} =, \text{ suppose, to } \frac{A^{n+1}}{y^n}.$$

Multiply both sides of the equation by $\dot{y}$ and take the fluents,

$$\text{then } \frac{b^2}{x^2} \times \frac{\dot{y}^2}{\dot{x}^2} = \frac{A^{n+1}}{(n-1)y^{n-1}}.$$

But $\dot{x} : \dot{y} :: b : v$, or $\frac{\dot{y}^2}{\dot{x}^2} = \frac{v^2}{b^2}$, when $y = r$,

therefore

therefore the correct fluents become

$$\frac{c^2}{2} - \frac{b^2}{2} \times \frac{\dot{y}^2}{x^2} = \frac{A^{n+1}}{n-1} \times \frac{y^{n-1} - r^{n-1}}{r^{n-1} y^{n-1}}; \text{ which gives}$$

$$\dot{x} = \frac{b \sqrt{(n-1)} \times r^{\frac{n-1}{2}} y^{\frac{n-1}{2}} \dot{y}}{\sqrt{(n-1)c^2 r^{n-1} y^{n-1} - 2A^{n+1} y^{n-1} + 2A^{n+1} r^{n-1}}}$$

But in Prop. 1. it was found, that if a body descended from an infinite height, and was acted upon by a force $= \frac{A^{n+1}}{y^n}$, the square of the velocity acquired at the distance r was $= \frac{2A^{n+1}}{(n-1)r^{n-1}}$;

let therefore $\frac{2mA^{n+1}}{(n-1)r^{n-1}} = c^2$, and then $A^{n+1} = \frac{(n-1)c^2 r^{n-1}}{2m}$.

Hence by substitution and reduction,

$$\dot{x} = \frac{b}{c} \times \frac{\sqrt{m} \times y^{\frac{n-1}{2}} \dot{y}}{\sqrt{\{(m-1)y^{n-1} + r^{n-1}\}}}$$

from whence the relation of x and y may be determined.

Let V = velocity of a body in a circle at the distance r from the centre of force, and which is acted upon by the same force as that which tends to the plane at the same distance from

it; then $\frac{V^2}{r} = \frac{A^{n+1}}{r^n}$, or $A^{n+1} = V^2 \times r^{n-1}$; from whence,

and the equation above, viz. $A^{n+1} = \frac{(n-1)c^2 r^{n-1}}{2m}$, it will

easily appear that $c = \frac{\sqrt{2m} \times V}{\sqrt{(n-1)}}$; therefore the above equation becomes

$$\dot{x} = \frac{b \sqrt{(n-1)}}{V \sqrt{2}} \times \frac{y^{\frac{n-1}{2}} \dot{y}}{\sqrt{\{(m-1)y^{n-1} + r^{n-1}\}}}$$

COR.

COR. 1. If c , and consequently $m = 0$, or which is the same thing, if the body be projected parallel to the horizon, then

$$\dot{x} = \frac{b \sqrt{(n-1)}}{V \sqrt{2}} \times \frac{y^{\frac{n-1}{2}} \dot{y}}{\sqrt{(r^{n-1} - y^{n-1})}}.$$

COR. 2. If $n = 3$, then the last equation in the proposition becomes

$$\dot{x} = \frac{b}{V} \times \frac{y \dot{y}}{\sqrt{\{(m-1)y^2 + r^2\}}}.$$

The fluent therefore being taken and corrected, by supposing x and y to begin together, we have

$$x = \frac{b}{V} \times \frac{\{(m-1)y^2 + r^2\}^{\frac{1}{2}}}{m-1} - r; \text{ therefore by reduction,}$$

$$y^2 = (m-1) \times \frac{V^2}{b^2} x^2 + \frac{2V}{b} r x = \frac{2V}{b} r x - (1-m) \times \frac{V^2}{b^2} x^2. \text{ an}$$

equation to a conic section; which will be a hyperbola, parabola, or ellipsis, according as m is greater, equal to, or less than unity; that is, according as the velocity with which the body is projected in a direction perpendicular to the horizon, is greater, equal to, or less than that acquired by falling from an infinite height.

If $(1-m) \times \frac{V^2}{b^2} = 1$, or $(1-m) \times V^2 = b^2$, then the equation becomes

$$y^2 = \frac{2r}{\sqrt{(1-m)}} \times x - x^2;$$

therefore the curve is a circle whose radius = $\frac{r}{\sqrt{(1-m)}}$.

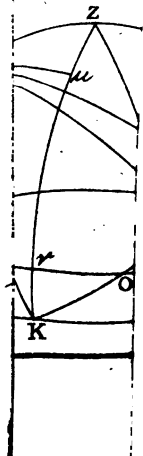
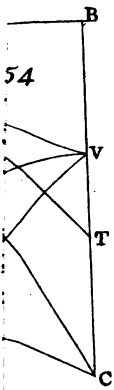
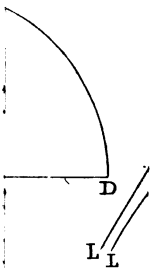
$$\text{But } V^2 r^{n-1} = \frac{(n-1)c^2 r^{n-1}}{2m}, \text{ therefore } m = \frac{(n-1)c^2}{2V^2} = \frac{c^2}{V^2} \text{ when } n=3$$

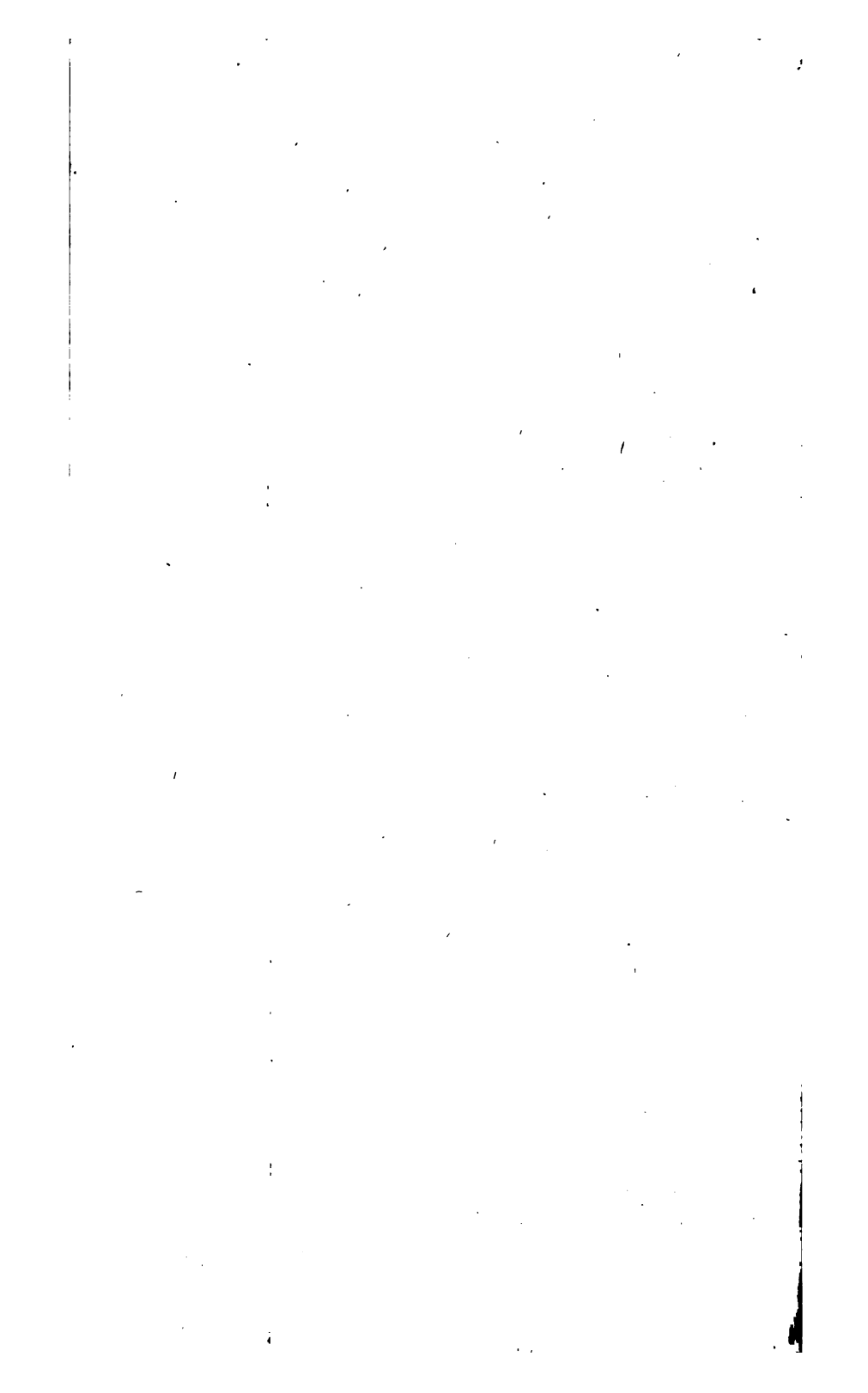
$$\text{hence the radius of the circle} = \frac{rV}{\sqrt{(V^2 - c^2)}}.$$

In the same manner it will appear, that the semi-transverse and conjugate axes

$$\text{of the hyperbola will be } r \times \frac{bV}{c^2 - V^2} \text{ and } r \times \frac{V}{\sqrt{(c^2 - V^2)}}.$$

and





and of the ellipsis $r \propto \frac{bV}{V^2 - c^2}$ and $r \propto \frac{V}{\sqrt{(V^2 - c^2)}}$;

and if the hyperbola be rectangular, its semi-axes $= r \propto \frac{V}{\sqrt{(c^2 - V^2)}}$.

COR. 3. If $n = 2$, the equation becomes

$$\dot{x} = \frac{b}{V\sqrt{2}} \times \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{(r - (1-m)y)}} = \frac{b}{V\sqrt{\{(1-m) \times 2\}}} \times \dot{y} \sqrt{\frac{y}{\frac{r}{1-m} - y}}.$$

But $\dot{x} = \dot{y} \sqrt{\frac{y}{\frac{r}{1-m} - y}}$ is the equation of a cycloid, the di-

ameter of whose generating circle $= \frac{r}{1-m} = r \times \frac{2V^2}{2V^2 - c^2}$;

m being $= \frac{c^2}{2V^2}$. If therefore $b^2 = 2V^2 \times (1-m) = 2V^2 - c^2$,

the curve will be a cycloid. Or if the value of x , in this cycloid, be taken to the corresponding value of x in the curve described,

as $V\sqrt{\{2(1-m)\}}$ to b , the curve may be easily constructed.

COR. 4. It was found above that $\frac{b^2}{2} \times \frac{\dot{y}^2}{\dot{x}^2} = \frac{A^{n+1}}{(n-1)y^{n-1}}$,

and at the vertex of the curve $\dot{y} = 0$, therefore the correct fluent becomes

$$\frac{b^2}{2} \times \frac{\dot{y}^2}{\dot{x}^2} = A^{n+1} \times \frac{d^{n-1} - y^{n-1}}{(n-1)d^{n-1}y^{n-1}}, \quad d \text{ being the value of } y$$

when $\dot{y} = 0$.

$$\text{If therefore } n = 0, \text{ then } \frac{b^2}{2} \times \frac{\dot{y}^2}{\dot{x}^2} = A \times \frac{d^{-1} - y^{-1}}{-d^{-1} \times y^{-1}}.$$

by reduction $\dot{x} = \frac{b}{\sqrt{(2A)}} \times \frac{-\dot{y}}{\sqrt{(d-y)}}$; hence

$$x = \frac{b}{\sqrt{(2A)}} \times \frac{(d-y)^{\frac{1}{2}}}{\frac{1}{2}} = \frac{b}{\sqrt{(\frac{1}{2}A)}} \times (d-y)^{\frac{1}{2}}, \text{ making } x=0,$$

when $y = d$.

The curve therefore is a parabola, whose axis is perpendicular to the plane, latus rectum $= \frac{b^2}{\frac{1}{2}A}$ and force $= A$, measured by the velocity generated in the time (1).

PROPOSITION V.

If a body revolve round a centre, and be acted upon by a force tending to that centre which varies as the n th power of the distance inversely, and whose quantity at an apse, the distance of which apse being r , is $= 1$; likewise, if another body, besides being subject to this force, be acted upon by an additional one, which varies inversely as the q th power of the distance, its value being c at the same distance r ; it is required to investigate a general expression for the ratio of the angular velocities of the two bodies when at the same distance from the centre of force.

Let the required ratio be that of $F : G$. Put $y =$ common distance, $p =$ perpendicular upon the tangent to the orbit described by the single force, and $p' =$ perpendicular upon the tangent to the other orbit, drawn from the common centre.

It is evident by proposition 1, that

$$F^2 : G^2 :: \frac{p^2 y^2}{(y^2 - p^2) y^3} : \frac{p'^2 y^2}{(y^2 - p'^2) y^3} :: \frac{p^2}{y^2 - p^2} : \frac{p'^2}{y^2 - p'^2}.$$

$$\text{But (by prop. 1st. and 3d.) } p^2 = \frac{\frac{m}{1-m} \times r^2 y^{n-1}}{\frac{r^{n-1}}{1-m} - y^{n-1}}, \text{ and}$$

$$p'^2 = \frac{Mr^2 y^{n-1}}{r^{n-1} + c \times \frac{n-1}{q-1} \times r^{q-1} y^{n-q} + \left(M-1-c \times \frac{n-1}{q-1} \right) \times y^{n-1}},$$

writing M here instead of m in the third proposition.

Hence, by substitution and reduction, $F^2 : G^2$

$$:: \frac{m}{(m-1)y^2 - mr^2 + r^{r-1}y^{3-n}} \cdot \frac{M}{\left(M-1-c \times \frac{n-1}{q-1} \right) \times y^2 + c \times \frac{n-1}{q-1} \times r^{q-1} y^{3-q} + r^{n-1} y^{3-n} - mr^2}.$$

But

But $m = \frac{n-1}{2} \times s^2$ and $M = \frac{(n-1)(1+c)}{2} \times S^2$ (writing S , for s in prop. 3.) therefore $F^2 : G^2$

$$\therefore \frac{s^2}{\{(n-1)s^2 - 2\} \times y^2 - (n-1)s^2 r^2 + 2r^{n-1} y^{3-n}} \\ : \frac{(1+c)S^2}{\{(n-1)(1+c)S^2 - 2c \cdot \frac{n-1}{q-1}\} \cdot y^2 + 2c \cdot \frac{n-1}{q-1} r^{q-1} y^{3-q} + 2r^{n-1} y^{3-n} - (n-1)(1+c)S^2 r^2}$$

which is general, whatever be the values of n and q , s and S .

COR. 1. If $q = 3$, then $F^2 : G^2$

$$\therefore \frac{s^2}{\{(n-1)s^2 - 2\} \times y^2 - (n-1)s^2 r^2 + 2r^{n-1} y^{3-n}} \\ : \frac{(1+c)S^2}{\{(n-1)(1+c)S^2 - 2c(n-1)\} \cdot y^2 - \{(n-1)(1+c)S^2 - c(n-1)\} \cdot r^2 + 2r^{n-1} y^{3-n}}$$

where it is evident that if we make $(n-1)(1+c)S^2 - c(n-1) = (n-1)s^2$, then the denominators will be equal, and therefore $F^2 : G^2 :: s^2 : (1+c)S^2$, which is a constant ratio; and therefore when $q = 3$, or when the additional force varies as the cube of the distance reciprocally, the angles, which lines drawn from the bodies to the centre of force, make with the line passing through the apse, are in a constant ratio to each other, whatever be their common distance (the condition mentioned being observed).

COR. 2. Because $(n-1)(1+c)S^2 - c(n-1) = (n-1)s^2$, therefore

$S^2 = \frac{s^2 + c}{1 + c}$. Let R = radius of curvature and v = velocity at the apse in the orbit described by one force; then the centripetal force at the distance r , $= \frac{v^2}{R} = \frac{v^2}{s^2 r}$, therefore

$$s^2 = \frac{R}{r} \text{ and } S^2 = \frac{s^2 + c}{1 + c} = \frac{R + rc}{r(1 + c)}; \text{ hence}$$

$$F^2 : G^2 :: \frac{R}{r} : \frac{R + rc}{r} :: R : R + rc, \text{ or}$$

$$F^2 : \frac{G^2 - F^2}{r} :: R : c = \frac{G^2 - F^2}{r F^2} \times R = \text{additional force at}$$

the apse. But because $q = 3$,

$y^3 : r^3 :: \frac{G^2 - F^2}{rF^2} \times R : \frac{G^2 - F^2}{F^2} \times \frac{Rr^2}{y^3} = \text{additional force at the distance } y, \text{ and the force in the first orbit at the same distance} = \frac{r^n}{y^n}$, therefore

the whole compound force $= \frac{r^n}{y^n} + \frac{G^2 - F^2}{F^2} \times \frac{Rr^2}{y^3}$.

COR. 3. If instead of (1), $\frac{v^2}{R}$ be supposed = force in the first orbit at the distance r , it will easily appear that the compound

force in the other orbit, at the distance y , will be $= \frac{r^n v^2}{R y^n} +$

$\frac{G^2 - F^2}{F^2} \times \frac{v^2 r^2}{y^3}$; the same expression that is found after a very different manner, Principia, Book 1st, Sect. 9th.

NOTE. (*Referred to in Page 45*).

Let VABD (fig. 159, pl. 8.) be a trajectory described by a body round a centre of force C, V an apse, and A the next following one; V being at a greater distance from the centre than A. It is evident that if the body were projected from A, at right angles to CA, and with the same velocity it had when it arrived there, it would accurately describe the arch AV, and have the same velocity at V that it first began with. For during the time of moving over any particle of the curve, the force acting upon the body, and the direction of the force are the same in both cases; hence the conclusion is clear. But if the body, instead of moving towards V, be projected in a contrary direction, at right angles to AC, and with the same velocity, an arch AB, equal and similar to AV, will be described; B being an apse, and CB = CV. Hence in the above equation y can have but two different values; but as these *may* be in opposite directions, two may be positive and two equal to them and negative. The other roots, if any, must either be impossible, or relate to such parts of the algebraical curve as have their concavity turned from the centre of force, being described by a centrifugal force, or such parts as are separated from that part in which the body moves; that is, it cannot be a curve of continued curvature, as that *must be* in which the body moves.

The same conclusion may be deduced immediately from the nature of the equation found above for determining the apsides. For making the equation of the

limits = 0, we have $y = \pm P \times \sqrt{\left(\frac{m}{m-1} \times \frac{q-9}{q-1}\right)}$. Whence it is

evident, that there can be no more than four roots, two positive and two negative, But to discover in every case the number of possible roots is a problem of considerable

siderable difficulty. DR. WARING has pointed out the method of doing this (*Meditationes Algebraicae*, Prob. 14.); but as his manner of writing is, in general, very concise, an easy investigation of the several conclusions there deduced may not be improper in this place.

It is well known, that if varying the coefficients of an equation two roots become equal, the next instant they will be impossible, and immediately before becoming equal they will be real and unequal.

This being granted, let it be proposed to find the number of possible roots in the equation $x^n - Ax^m - B = 0$, which is an equation of the same kind with that for determining the apsidæ. First, find the equation of limits, and make it $= 0$, viz.

$$nx^{n-1} - mAx^{m-1} = 0, \text{ then } x = \left(\frac{m}{n}\right)^{\frac{1}{n-m}} \times A^{\frac{1}{n-m}};$$

hence x has two values, if n and m be both even numbers, and $-A$ a negative number, wherefore the number of real roots in the given equation cannot exceed four.

Multiply the first equation by n , and the second by x , and take the difference

$$\text{of the products, then } (m-n) \times Ax^m - nB = 0; \text{ from whence } x = \left(\frac{n}{m-n}\right)^{\frac{1}{m}} \times \left(\frac{B}{A}\right)^{\frac{1}{m}}, \text{ which will give two other limits, if } -B \text{ be positive, and the rest as above.}$$

If the given equation have two equal roots, they will coincide both with the first and second values of x just found; therefore these values must be equal to each other; and conversely, if these values be equal, the given equation must have two equal roots; and if the condition abovementioned take place, viz. that n and m , be both even numbers, $-A$ negative, and $-B$ positive, then there will be two pairs of equal roots. Make them equal to each other, then by reduction it will easily appear that

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m} = 0; \text{ hence, by making this ex-}$$

pression positive, or negative, according to circumstances, the number of possible roots may be obtained.

1. If n be an even number, and $-B$ negative, it is evident that the equation has two real roots. If n be even and m odd, and $-B$ affirmative, and at the same time

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m} \text{ be negative, then there will be two}$$

possible roots; otherwise none. For the first part of the above expression is in this case negative, and the other part positive; but the first part must be greater than the second, if the roots be real; because if it vanish, all the roots are impossible; hence the conclusion is clear.

Let n and m , be even numbers, and $-A$, and $-B$ positive quantities; it is evident that the equation can have no real roots; for in this case no quantity substituted for x can make the result $= 0$. But if $-B$ be affirmative, and $-A$ negative, and at the same time

$$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m} \text{ be affirmative then there will be}$$

our real roots, otherwise none. For, from what has been said above, there will be two pairs of equal roots when the above expression $= 0$, wherefore by making it affirmative there will be four real roots. That the first part of the expression must be greater than the second will be evident from observing, that if $A = 0$, then all the roots will be impossible.

2 If both n and m be odd numbers, and

$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m}$ affirmative, then there will be three

possible roots; otherwise only one. Because n is an odd number there must be one real root: and if the above expression $= 0$, there will be other two roots equal. But if A and B be taken such as to make it affirmative, there will be other two real roots. For $n - m$ is an even number, therefore the first part of the expression is positive; and if $A = 0$, there will be only one real root, and the above expression would in that case be negative. Therefore, &c.

If n be odd, and m even, and A and B , have the same sign, then the equation will have only one real root; this requires no proof. But if they have different signs, and at the same time

$\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n - \frac{(n)^m}{(m)^m} \times B^{n-m}$ have a contrary sign to B , then the

equation will have three real roots; otherwise but one. If the equation have this form, viz.

$x^n + Ax^m - B = 0$, then $\frac{(m-n)^{n-m}}{(n)^{n-m}} \times A^n$ will be positive, and it must

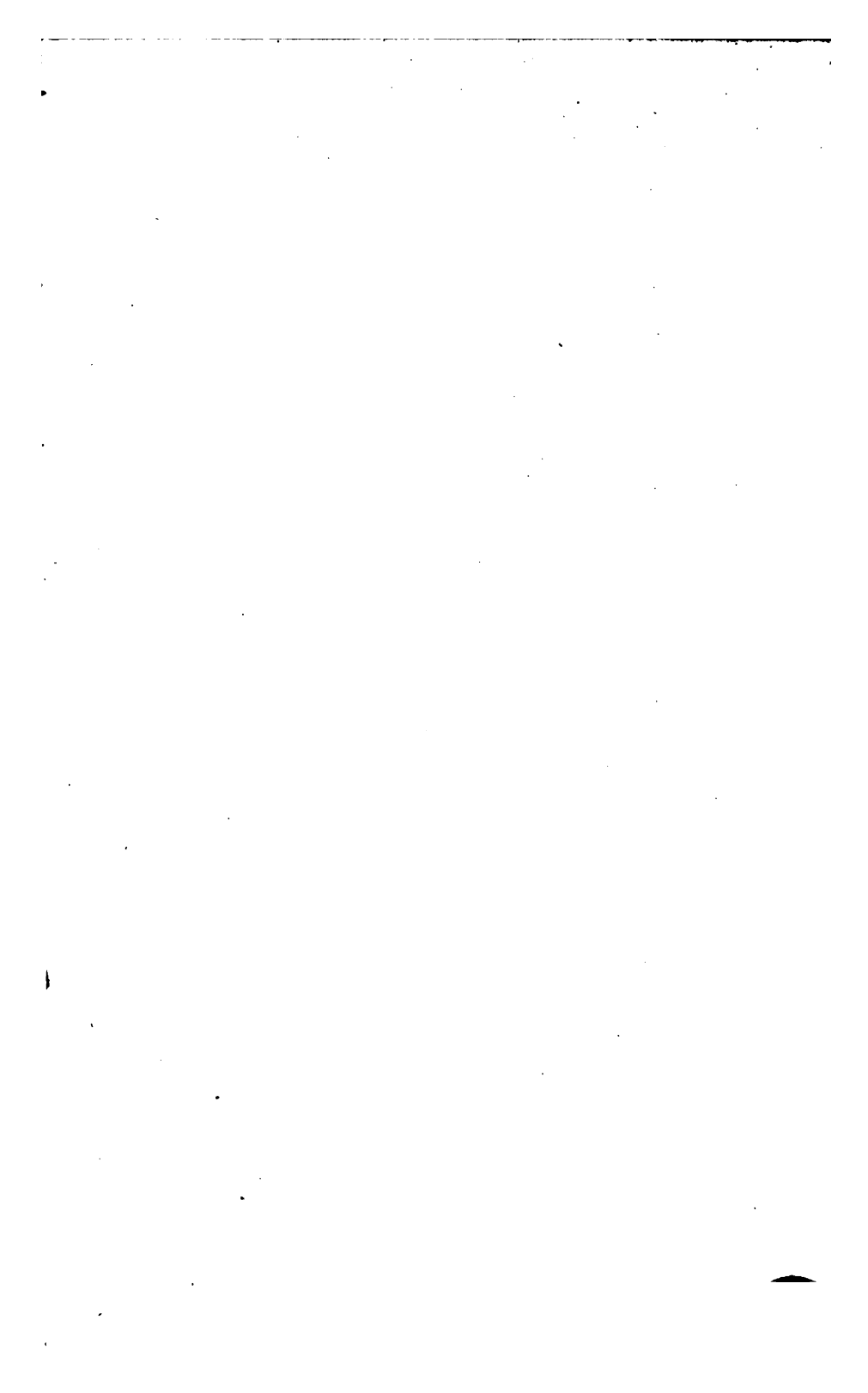
be greater than $\frac{(n)^m}{(m)^m} \times B^{n-m}$, for reasons given above; therefore the whole is positive.

But if it be of this form, viz. $x^n - Ax^m + B = 0$, then the first part of the general expression is negative, and the second part positive, wherefore the result must be negative.

N.B. All the figures referred to in this Article are to be found in plate VIII. (A) and not in plate VIII.

END OF VOL. I.

Glendinning, Printer, 25, Hatton Garden.



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